Journal of Statistical Physics, Vol. 83, Nos. 3/4, 1996

Metastability and Nucleation for the Blume–Capel Model. Different Mechanisms of Transition

Emilio N. M. Cirillo[1] **and Enzo Olivieri**[2]

Received May 1995; final September 28, 1995

We study metastability and nucleation for the Blume–Capel model: a ferromagnetic nearest neighbor two-dimensional lattice system with spin variables taking values in $\{-1, 0, +1\}$. We consider large but finite volume, small fixed magnetic field h, and chemical potential λ in the limit of zero temperature; we analyze the first excursion from the metastable -1 configuration to the stable $+1$ configuration. We compute the asymptotic behavior of the transition time and describe the typical tube of trajectories during the transition. We show that, unexpectedly, the mechanism of transition changes abruptly when the line $h = 2\lambda$ is crossed.

KEY WORDS: Blume–Capel model; stochastic dynamics; metastability; nucleation.

1. INTRODUCTION

Metastability is a relevant phenomenon for thermodynamic systems close to a first order phase transition.

Let us start from a given pure equilibrium phase in a suitable region of the phase diagram and change the thermodynamic parameters to values corresponding to a different equilibrium phase; then, in particular experimental situations, the system, instead of undergoing a phase transition, can still remain in a "wrong" equilibrium, far from the "true" one but actually close to what the equilibrium would be at the other side of the transition. This apparent equilibrium, often called the "metastable state,"

[1] Dipartimento di Fisica dell'Università di Bari and Istituto Nazionale di Fisica Nucleare, Sezione di Bari, I-70126 Bari, Italy; E-mail: cirillo@axpba0.ba.infn.it.

[2] Dipartimento di Matematica, II Università di Roma Tor Vergata, 00133 Rome, Italy; E-mail: olivieri@mat.utovrm.it.

0022-4715/96/0500-0473$09.50/0

persists until an external perturbation or some spontaneous fluctuation leads the system to the stable equilibrium.

For general review on metastability with particular attention to rigorous results see refs. 17 and 18.

There are strong arguments leading to the conclusion that metastability can neither be included in the scheme of the Gibbsian formalism, which is confined to the description of the genuine stable equilibrium states,[11] nor can it be directly described using extrapolation beyond the condensation point, because of the presence of an essential singularity of the free energy (see the fundamental result due to Isakov.)[8]

Metastability is a genuine dynamical phenomenon. Its description, on one hand, is of basic importance from the point of view of fundamental physics, and on the other hand it poses interesting new mathematical problems.[4, 15, 16]

Since a general approach to nonequilibrium statistical mechanics is still missing, a crucial role is played by particular models of microscopic dynamics. It is remarkable that, quite recently, rigorous results have been deduced in this field by analyzing, in particular, the geometry of the condensation nuclei as well as the possible coalescence between droplets. Notice that these aspects were totally absent in previous theories such as the socalled classical theory of nucleation.[17]

In recent years much progress has been made in the understanding of the phenomenon of metastability in the framework of Glauber dynamics. By Glauber dynamics we mean a stochastic time evolution of a lattice spin system (in continuous or discrete time) whose elementary process is a single spin change and which is *reversible* (namely it satisfies the detailed balance condition) with respect to the Gibbs measure corresponding to a given Hamiltonian. There is a certain freedom in choosing a particular dynamics satisfying the above requirements. A typical choice, which actually we will make in the present paper, is called "Metropolis dynamics" [see Eq. (2.6) below].

The case of the standard Ising model (spin $+1$ *or* -1, ferromagnetic nearest neighbor interaction), often referred to as the *stochastic Ising model* or *kinetic Ising model*, has been analyzed, in two dimensions in ref. 12 in connection with relaxation to equilibrium for arbitrarily large (and even infinite) systems close to the first-order phase transition.

In a quite complete treatment, Neves and Schonmann[13, 14] analyzed, in the framework of the "pathwise approach to metastability" introduced some years ago in ref. 4, the phenomenon of nucleation for large but finite volume and small magnetic field in the zero temperature limit.

Schonmann,[19] using an argument based on reversibility, described in detail the typical escape paths.

Other asymptotic regimes, very interesting from a physical point of view and mathematically much more complicated, are considered in refs. 20 and 21.

In the same asymptotic regime as in ref. 13, different Hamiltonians have been considered in refs. 9 and 10, where it has been shown that the typical path followed during the growth of the stable phase in general is not of Wulff type. Here by Wulff (shape) we mean the equilibrium shape of a droplet at zero temperature, namely the shape minimizing the surface energy for fixed volume. This non-Wulff growth seems to be an interesting phenomenon in the description of crystal growth.

Let us now try to explain the nature of the mathematical difficulties related to our problem. We notice that in the above-mentioned asymptotic regime the behavior is similar to the one described by Freidlin and Wentzell in their analysis of small random perturbations of dynamical systems: the system typically performs random oscillations around the local minima of the energy and sometimes it goes against the drift following some preferential ways. In particular, it is interesting to characterize the typical tube of trajectories during the first excursion from the metastable to the stable equilibrium. This first excursion can be seen as an escape from a sort of generalized basin $\mathcal{G}$ of attraction of the metastable equilibrium. It turns out that many local equilibria are contained in $\mathcal{G}$ and this more general situation goes beyond the approach developed by Freidlin and Wentzel,[7] who were able to give a full description of the typical tube of escape only for the case of a domain D completely attracted by a unique stable point.

In our more general case, as we will see, new interesting phenomena take place involving a sort of "temporal entropy." These phenomena are taken into account in refs. 15 and 16, where a complete description of the typical tube of escape is given in general.

For attractive short-range systems the main feature of the transition appears to be the formation of a critical nucleus with suitable shape and size. This critical droplet results from a competition between the bulk energy favoring growth and the surface energy favoring contraction. Only for large sizes and for particular shapes will the droplet tend to grow.

The present paper is devoted to the study of metastability and nucleation in the framework of a dynamical Blume–Capel model. It is a two-dimensional spin system where the single spin variable can take three possible values: $-1, 0, +1$. It was originally introduced to study the ^3He–^4He phase transition.

One can think of it as a system of particles with spin. The value $\sigma_x = 0$ of the spin at the lattice site x will correspond to absence of particles (a *vacancy*), whereas the values $\sigma_x = +1, -1$ will correspond to the presence at x of a particle with spin $+1$, -1, respectively. The formal Hamiltonian is given by

$$H(\sigma) = J \sum_{\langle x, y \rangle} (\sigma_x - \sigma_y)^2 - \lambda \sum_x \sigma_x^2 - h \sum_x \sigma_x \tag{1.1}$$

where λ and h are two real parameters, having the meaning of the chemical potential and the external magnetic field, respectively; J is a real, positive constant (ferromagnetic interaction) and $\langle x, y \rangle$ denotes a generic pair of nearest neighbor sites in $\mathbf{Z}^2$.

In the following we will consider the system enclosed in a two-dimensional torus Λ. Let $-\underline{1}$, $\underline{0}$, and $+\underline{1}$ denote the configurations with all the spins in Λ equal to -1, 0, $+1$, respectively. The structure of the set of ground states corresponding to different values of λ and h will be discussed in Section 2. Now we only note that it is immediate to see that for $\lambda = h = 0$ the configurations $-\underline{1}$, $\underline{0}$, and $+\underline{1}$ are the only ground states. It has been shown, using Pirogov–Sinai theory, that this phase transition persists at positive temperature $T = 1/\beta$ in the thermodynamic limit.[1–3, 5]

We will use as dynamics the Metropolis algorithm, in which the typical time needed to overcome an energy barrier ΔH is of order $\exp(\beta \Delta H)$. It will be defined in detail in the next section.

We are interested in the case in which λ and h are very small but fixed, the volume is large and fixed, and T is very small; namely, we move in the vicinity of the triple point $h = \lambda = T = 0$. In particular we will consider the region $h > \lambda > 0$, where the most interesting phenomena take place. The stable equilibrium, namely the absolute minimum of the energy, in this case is $+\underline{1}$ and we suppose to start with the system in the configuration $-\underline{1}$. We want to describe the first excursion between $-\underline{1}$ and $+\underline{1}$. It turns out that in the above region a direct interface between pluses and minuses is unstable and its appearance and resistance are very unlikely.

The main effect which surprisingly and unexpectedly shows up is that two different mechanisms of transition between $-\underline{1}$ and $+\underline{1}$ take place for different values of the parameters λ, h. More precisely, for $0 < 2\lambda < h$ the transition takes place via the formation of a suitable critical droplet of zeros that keeps growing until it covers the whole volume. Subsequently, from the intermediate zero phase we have the nucleation of a critical droplet of plus spins driving eventually the system to the stable $+\underline{1}$ phase.

Conversely, for $0 < \lambda < h < 2\lambda$, the plus phase is created directly from the minus phase via the formation of a suitable critical nucleus, a sort of "picture frame" (see Fig. 10), containing in its bulk the plus spins with a thin layer of zeros separating the interior pluses from the sea of minuses. We want to stress that the line $h = 2\lambda$, where this abrupt variation of the mechanism of nucleation takes place, has no meaning from the "static" point of view of the Gibbs states. The reason is that we are analyzing a region of the configuration space very unlikely at the equilibrium; but, on the other hand, this region and the form of the "energy landscape" on it play important roles in the relaxation from metastability.

A first result that we obtain in the present paper refers to the computation of the asymptotic behavior for small temperatures, of the transition time (the lifetime of the metastable state). Then we pass to the characterization of typical trajectories during the transition; we specify the geometrical sequence of droplets as well as the order of magnitude of the necessary time fluctuations during the growth of the critical nucleus both for $h < 2\lambda$ and for $h > 2\lambda$. To do this we exploit some general results contained in ref. 15. The model-dependent part of the work consists in the solution of a well-specified sequence of variational problems. The main difficulty is the determination of the "minimal global saddle" between $-\underline{1}$ and $+\underline{1}$ and of the set $\mathscr{G}$, the generalized basin of attraction of $-\underline{1}$. From this we will single out an optimal nucleation mechanism. We will analyze the energy landscape precisely to exclude all the other possible mechanisms of transition. In particular we will show that any form of coalescence will be reduced in probability with respect to the optimal nucleation mechanism.

The paper is organized in the following way: Section 2 contains definitions and results. In particular we state Theorem 1 concerning the asymptotics of the escape time. In Section 3 we describe the local minima of the energy. In Section 4 we discuss supercriticality or subcriticality of droplets, namely we determine their tendency to grow or shrink. In Section 5 we prove a basic result on the height of different global saddles. In Section 6 we define the set $\mathscr{G}$ and find the minima of the energy in its boundary $\partial\mathscr{G}$. In Section 7 we describe the typical tube of trajectories during the first excursion; then, using as preliminary results the propositions contained in the previous section, we conclude the proof of Theorem 1; finally we state and prove Theorem 2, which refers to the typical tube. Section 8 contains the conclusions. An appendix contains an explicit proof of a useful result about recurrence properties of a general class of Markov chains.

2. DEFINITIONS AND RESULTS

We start by describing the model that we want to study. The configuration space is

$$\Omega_\Lambda = \{-1, 0, +1\}^\Lambda \tag{2.1}$$

where $\Lambda = \Lambda_L$ is a two-dimensional torus (a square with periodic boundary conditions) of side L.

A configuration σ is a function

$$\sigma: \Lambda \to \{-1, 0, +1\}$$

The energy associated to the configuration σ is given by

$$H(\sigma) = J \sum_{\langle x, y \rangle \subset \Lambda} (\sigma_x - \sigma_y)^2 - \lambda \sum_{x \in \Lambda} \sigma_x^2 - h \sum_{x \in \Lambda} \sigma_x \tag{2.2}$$

where $\langle x, y \rangle$ denotes a generic pair of nearest neighbor sites in Λ and we suppose $0 < \lambda < h < J$. We also introduce the restriction

$$\lambda < \frac{2J}{2a^2 + a - 1}$$

where $a := h/\lambda$, the meaning of this condition will be clear later on [see (3.17)].

The *Gibbs measure* in the torus Λ is given by

$$\mu_\Lambda = \frac{\exp(-\beta H(\sigma))}{Z_\Lambda} \tag{2.3}$$

where β represents the inverse temperature and Z_Λ is the normalization factor, called the *partition function.*

We describe now the structure of the ground states corresponding to the different values of our parameters λ and h.

Let $-\underline{1}$, $\underline{0}$, and $+\underline{1}$ denote the configurations with all the spins in Λ equal to $-1, 0, +1$, respectively. We have:

for $\lambda = h = 0$	the ground state is three times degenerate, the configurations minimizing the energy are $-\underline{1}$, $\underline{0}$ and $+\underline{1}$
for $h > 0$ and $h > -\lambda$	the ground state is $+\underline{1}$
for $h < 0$ and $h < \lambda$	the ground state is $-\underline{1}$
for $\lambda < 0$ and $\lambda < h < -\lambda$	the ground state is $\underline{0}$

For $h = 0$, $\lambda > 0$: $+\underline{1}$, $-\underline{1}$ coexist. For $h = \lambda < 0$: $-\underline{1}, \underline{0}$ coexist. For $h = -\lambda > 0$: $+\underline{1}, \underline{0}$ coexist. These results are summarized in Fig. 1, where the coexistence lines are shown.

We now introduce a dynamics in our model. It will be a discrete–time Glauber dynamics, namely a Markov chain with state space given by Ω_Λ, with the following properties:

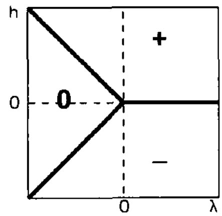


Fig. 1. Ground states for the Blume–Capel model.

1. The allowed transitions are between *nearest neighbor configurations*, namely pairs ξ and η of configurations differing only in one spin: $\xi = \eta^{x,b}$, with

$$\eta^{x,b}(y) := \begin{cases} \eta(y) & \forall y \in \Lambda, \quad y \neq x \\ b & \text{for} \quad y = x \end{cases} \tag{2.4}$$

with $b \in \{-1, 0, +1\}$.

2. It is *reversible* w.r.t. the Gibbs measure μ_Λ for the Blume–Capel model; namely the transition probabilities $P(\sigma, \sigma')$ of the chain satisfy

$$\mu_\Lambda(\sigma)\, P(\sigma, \sigma') = \mu_\Lambda(\sigma')\, P(\sigma', \sigma) \tag{2.5}$$

Our choice is the so-called Metropolis algorithm, where the transition probabilities, for pairs of different configurations σ, η, are defined as

$$P(\sigma, \eta) := \begin{cases} \dfrac{1}{2\,|\Lambda|}\, e^{-\beta[H(\eta) - H(\sigma)]^+} & \sigma, \eta \text{ nearest neighbors} \\ 0 & \text{otherwise} \end{cases} \tag{2.6}$$

where

$$a^+ := \begin{cases} a & \text{if} \quad a \geqslant 0 \\ 0 & \text{if} \quad a < 0 \end{cases} \qquad \forall a \in \mathbf{R} \tag{2.7}$$

The *space of trajectories* of the process is

$$\Xi := (\Omega_\Lambda)^{\mathbf{N}}$$

An element in Ξ is denoted by ω; it is a function

$$\omega\colon \mathbf{N} \to \Omega_\Lambda$$

We often write $\omega = \sigma_0, \sigma_1, \ldots, \sigma_t, \ldots$.

We will call *path* an *allowed* trajectory, namely: $\sigma_0, \sigma_1, ..., \sigma_t, ...$ is a path iff σ_j and σ_{j+1} $\forall_j$ are *connected* in the sense that $\sigma_{j+1} = \sigma_j^{x,b}$ for some $x \in \Lambda$ and $b \in \{-1, 0, +1\}$. We use the notation $\omega: \sigma \to \eta$ to denote a path ω joining σ to η.

A path $\omega = \sigma_0, \sigma_1, ..., \sigma_n$ is called *downhill* (*uphill*) iff $H(\sigma_{j+1}) \leqslant H(\sigma_j)$ $[H(\sigma_{j+1}) \geqslant H(\sigma_j)]$ $\forall_j = 0, 1, ..., n-1$. We will use the convention that a downhill path can (and will) end only in a local minimum.

A set Q of configurations, $Q \subset \Omega_\Lambda$, is said to be *connected* iff for every pair of configurations $\sigma, \eta \in Q$, $\exists$ a path $\omega: \sigma \to \eta$ such that $\omega \subset Q$.

We say that a configuration σ is *downhill connected* to η iff there exists a downhill path $\omega: \sigma \to \eta$.

We will denote by M the set of all the locally stable configurations, namely the set of all the local minima of the energy. More precisely: $\sigma \in M$ iff for every $x \in \Lambda, b \in \{-1, 0, +1\}$ the corresponding increment in energy, given by

$$\Delta_{x,b} H(\sigma) := H(\sigma^{x,b}) - H(\sigma) \tag{2.8}$$

is positive.

It is easy to see that in our model with the choice of the parameters J, h, λ that we have made, the quantity $\Delta_{x,b} H(\sigma)$ will be always nonzero and this somehow simplifies some arguments.

Given $Q \subset \Omega_\Lambda$ we define the (outer) boundary ∂Q of Q as the set

$$\partial Q := \{\sigma \notin Q: \exists \sigma' \in Q: P(\sigma', \sigma) > 0\}$$

namely

$$\partial Q := \{\sigma \notin Q: \exists x \in \Lambda, b \in \{-1, 0, +1\} \text{ such that } \sigma' = \sigma^{x,b} \in Q\} \tag{2.9}$$

We denote by $U = U(Q)$ the set of all the minima of the energy on the boundary ∂Q of Q:

$$U(Q) := \{\zeta \in \partial Q: \min_{\sigma \in \partial Q} H(\sigma) = H(\zeta)\} \tag{2.10}$$

and we define $H(U(Q)) := H(\zeta)$ with $\xi \in U(Q)$.

We denote by $F = F(Q)$ the set of all minima of the energy on Q:

$$F(Q) := \{\zeta \in Q: \min_{\sigma \in Q} H(\sigma) = H(\zeta)\} \tag{2.11}$$

Given a stable state $\sigma \in M$, i.e., a local minimum for the energy, we define the following *basins* for σ:

(i) The *wide basin of attraction of* σ:

$$\hat{B}(\sigma) := \{\zeta: \exists \text{ downhill path } \omega: \zeta \to \sigma\} \tag{2.12}$$

(ii) The *basin of attraction* of σ given by

$$B(\sigma) := \{\zeta: \text{every downhill path starting from } \zeta \text{ ends in } \sigma\} \tag{2.13}$$

$B(\sigma)$ can be seen as the usual basin of attraction of σ with respect to the $\beta = \infty$ dynamics.

(iii) $\bar{B}(\sigma) =$ the *strict basin of attraction of* σ given by

$$\bar{B}(\sigma) := \{\zeta \in B(\sigma): H(\zeta) < H(U(B(\sigma)))\} \tag{2.14}$$

We introduce now the useful notion of cycle. A connected set A which satisfies

$$\max_{\sigma \in A} H(\sigma) = \bar{H} < \min_{\zeta \in \partial A} H(\zeta) = H(U(A))$$

is called a *cycle*. Notice that every local minimum for the energy is a (trivial) cycle.

The following simple properties of cycles are true. Their proof is immediate (see, for instance, ref. 15).

- Given a state $\bar{\sigma} \in \Omega_\Lambda$ and a real number c, the set of all σ's connected to $\bar{\sigma}$ by paths with energy always below c either coincides with Ω_Λ or it is a cycle A with

$$H(U(A)) \geqslant c$$

- Given two cycles A_1, A_2, either (i) $A_1 \cap A_2 = \emptyset$ or (ii) $A_1 \subset A_2$ or, vice versa, $A_2 \subset A_1$.

We give now some more definitions: a cycle A for which there exists $\eta^* \in U(A)$ downhill connected to some point σ in A^c is called *transient*; given a transient cycle A the points η^* downhill connected to A^c are called *minimal saddles*. The set of all the minimal saddles of a transient cycle A is denoted by $\mathcal{S}(A)$.

A transient cycle A such that $\exists \bar{\sigma} \notin A$ with $H(\bar{\sigma}) < H(F(A))$, and there exist $\eta^* \in \mathcal{S}(A)$ and a path $\omega: \eta^* \to \bar{\sigma}$ a *below* η^* [namely $\forall \sigma \in \omega$: $H(\sigma) < H(\eta^*)$] is called *metastable*.

For each pair of states σ, $\eta \in \Omega_\Lambda$ we define their minimal saddle $\mathcal{S}(x, y)$ as the set of states corresponding to the solution of the following minimax problem: let, for any path ω,

$$\hat{H}(\omega) := \max_{\zeta \in \omega} H(\zeta), \qquad \bar{H}_{\sigma, \eta} := \min_{\omega: \sigma \to \eta} \hat{H}(\omega)$$

Find

$$\mathscr{S}(\sigma, \eta) := \{\zeta\colon H(\zeta) = \bar{H}_{\sigma,\eta};\ \exists\omega\colon \sigma \to \eta, \omega \ni \zeta, \hat{H}(\omega) = \bar{H}_{\sigma,\eta}\}$$

One immediately verifies that a strict basin of attraction of a local minimum is a transient cycle. This case corresponds to a "one-well" structure. More general cases involve the presence of "internal saddles" and correspond to a "several wells" situation.

Given any set of configurations $A \subset \Omega_\Lambda$, we use τ_A to denote the *first hitting time* to A:

$$\tau_A := \inf\{t \geqslant 0\colon \sigma_t \in A\} \tag{2.15}$$

We use $P_\eta(\cdot)$ to denote the probability distribution over the process starting at $t = 0$ from the configuration η.

We are interested in dynamics at very low temperatures. Namely, we will discuss the asymptotic behavior, in the limit $\beta \to \infty$, of typical paths of the first escape from $-\underline{1}$ to $+\underline{1}$ for fixed h, λ, and Λ.

Let us now better clarify the asymptotic regime in which we will operate: the volume $|\Lambda|$, the magnetic field h, and the chemical potential Λ are fixed and we consider asymptotic estimates for β very large. This regime was studied in the case of the standard Ising model in 2D by Neves and Schonmann,[13] where the point of view of the *pathwise approach to metastability*, introduced in ref. 4, was assumed.

One can, for instance, take λ very small, $h = a\lambda$, a fixed positive number, $|\Lambda|$ of order, say, $1/h^2$, and β of order $1/h^5$; physically this corresponds to a regime in which, at the equilibrium, the energy dominates w.r.t. the entropy.

In the above-described situation the qualitative behavior of our stochastic time evolution can be described as follows: the system will spend the majority of the time in the local minima of the energy. Sometimes it escapes from them, but there is always a natural tendency to follow a downhill path and an occasional, random and rather unprobable, uphill move.

An important role will be played by the saddles separating different "basins of attraction" (or generalized basins of attraction, see below) w.r.t. the $\beta = \infty$ dynamics.

We will see that the local minima will correspond to particular geometric shapes that will be called *plurirectangles* (see Fig. 9); we will analyze, in particular, the *special* saddles between "contiguous" local minima (see Lemma 5.1).

A *global saddle point* is any configuration

$$\bar{\sigma} \in \mathscr{S}(-\underline{1}, +\underline{1}) =: \mathscr{P}$$

In Section 6 we will see that the set of these global minima $\mathscr{P}$ are substantially different according to the values of the parameters λ and h.

1. For $h < 2\lambda$ we distinguish two cases:

(a) If $\delta := l^* - (2J - (h - \lambda))/h < (h + \lambda)/2h$, $\mathscr{P}$ is of the form $\mathscr{P}_{1,a}$ given in Fig. 32 [the two critical lengths l^* and M^* are defined in (3.12) and (3.15)]; namely it contains a "droplet" with external rectangle given by a square of side $l^* + 2$; the internal shape is given by a rectangle with sides l^*, $l^* - 1$, at a distance one from the external rectangle and with a unit-square protuberance attached to the longest "free" side; the internal shape is full of pluses; and the spins lying outside the exterior rectangle are minuses; finally, between the interior shape and the external rectangle there are zeros.

(b) If $\delta > (h + \lambda)/2h$, $\mathscr{P}$ is of the form $\mathscr{P}_{1,b}$ depicted in Fig. 32. Now $\mathscr{P}_{1,b}$ is similar to $\mathscr{P}_{1,a}$, but now the external rectangle has sides $l^* + 1$, $l^* + 2$ and internally we have a square with sides $l^* - 1$ with a unit-square protuberance attached to the shortest "free" side.

2. For $h > 2\lambda$, $\mathscr{P}$ is of the form $\mathscr{P}_2$ given in Fig. 32; namely it is given by a rectangle of sides M^* and $M^* - 1$, with a unit-square protuberance attached to one of its longest sides, full of zeros in a "sea" of minuses.

We set

$$\Gamma := H(\mathscr{P}) - H(-\underline{1}) \tag{2.16}$$

Let us now summarize our main results.

We shall prove that the first excursion from $-\underline{1}$ to $+\underline{1}$ typically passes through a configuration from $\mathscr{P}$ and the time needed for this to happen is of the order $\exp(\beta\Gamma)$; this is the content of Theorem 1 that we are now going to state.

Theorem 2 will characterize the typical trajectories of the first excursion. The proof of Theorems 1 and 2 and even the statement of Theorem 2 will need many more definitions and propositions. For this reasons they will be postponed to Section 7.

Theorem 1 is based, in particular, on Propositions 4.1–4.3 given in Section 4. These propositions refer to the tendency of a given minimum η of the energy to evolve toward $+\underline{1}$ or to $-\underline{1}$, namely they establish under which conditions a *droplet is supercritical* or not.

It will be crucial to introduce a sort of generalized basin of attraction of $-\underline{1}$. Indeed we will reduce the proof of Theorem 1 to finding a certain set $\mathscr{G}$ of configurations satisfying suitable properties. In order to explicitly construct this set $\mathscr{G}$, we will need the results contained in Propositions 4.1–4.3. This construction will be achieved in Section 6.

Theorem 1. Let $\bar{\tau}_{-\underline{1}}$ be the last instant in which $\sigma_t = -\underline{1}$ before $\tau_{+\underline{1}}$:

$$\bar{\tau}_{-\underline{1}} := \max\{t < \tau_{+\underline{1}} : \sigma_t = -\underline{1}\} \tag{2.17}$$

Let

$$\bar{\tau}_{\mathscr{P}} := \min\{t > \bar{\tau}_{-\underline{1}} : \sigma_t = \mathscr{P}\} \tag{2.18}$$

For every $\varepsilon > 0$: (i)

$$\lim_{\beta \to \infty} P_{-\underline{1}}(\bar{\tau}_{\mathscr{P}} < \tau_{+\underline{1}}) = 1 \tag{2.19}$$

and (ii)

$$\lim_{\beta \to \infty} P_{-\underline{1}}(\exp[\beta(\Gamma - \varepsilon)] < \tau_{+\underline{1}} < \exp[\beta(\Gamma + \varepsilon)]) = 1 \tag{2.20}$$

3. LOCAL MINIMA OF THE HAMILTONIAN $H(\sigma)$

In this section we want to analyze the geometrical structure of the local minima of the energy.

For any configuration $\sigma \in \Omega_\Lambda$ we denote by $c^+(\sigma)$, $c^-(\sigma)$, and $c^0(\sigma)$ the union of all closed unit squares centered at sites $x \in \Lambda$ with $\sigma(x)$ respectively equal to $+1$, -1 and 0. The $c^+(\sigma)$, $c^-(\sigma)$, and $c^0(\sigma)$ decompose into maximal connected components $c_j^{+,0,-}$, $j = 1,..., k^{+,0,-}$.

The centers of $c_j^{+,0,-}$ form a $*$-cluster in the sense of site percolation, namely they are maximally connected components in the sense of the next nearest neighbors. The $c_j^{+,0,-}$ will be simply called *clusters*.

To any such $c_j^{+,0,-}$ we assign its *rectangular envelope* defined as the minimal closed rectangle $R(c_j^{\pm,0})$ containing it; if none of the rectangles $R(c_j^{+,0})$ is winding around the torus, we call the corresponding configuration *acceptable.*

Let σ be an acceptable configuration; we denote by $\gamma_j^{+,0}$ the boundary of $c_j^{+,0}$ $\forall j \in \{1,..., k^{+,0}\}$; the internal component $\check{\gamma}_j^{+,0}$ of the boundary is defined as follows: let s be a unit segment of the dual lattice $\mathbf{Z}^2 + (1/2, 1/2)$ belonging to $\gamma_j^{+,0}$; we say that $s \in \check{\gamma}_j^{+,0}$ if and only if all the paths joining nearest neighbor sites of Λ and starting from the site adjacent to s and not in $c_j^{+,0}$, necessarily reach a site in $c_j^{+,0}$ before touching the cluster c_j^- winding around the torus. The external component $\hat{\gamma}_j^{+,0}$ of the boundary of $c_j^{+,0}$ is defined as $\gamma_j^{+,0} \setminus \check{\gamma}_j^{+,0}$. Of course $\check{\gamma}_j^{+,0}$ can be empty.

In order to construct the local minima of the Hamiltonian we first prove that direct $+-$ interfaces cannot exist in such configurations; in Fig. 2 we analyze the interaction of a minus spin with its neighboring sites.

$$\begin{matrix} & 0 & \\ + & - & 0 \\ & 0 & \end{matrix} \quad \begin{cases} 0 & -6J-(h-\lambda) \\ + & -4J-2h \end{cases} \qquad \begin{matrix} & 0 & \\ + & - & 0 \\ & + & \end{matrix} \quad \begin{cases} 0 & -8J-(h-\lambda) \\ + & -8J-2h \end{cases}$$

$$\begin{matrix} & 0 & \\ + & - & 0 \\ & - & \end{matrix} \quad \begin{cases} 0 & -4J-(h-\lambda) \\ + & -3J-2h \end{cases} \qquad \begin{matrix} & 0 & \\ + & - & + \\ & + & \end{matrix} \quad \begin{cases} 0 & -10J-(h-\lambda) \\ + & -12J-2h \end{cases}$$

$$\begin{matrix} & 0 & \\ + & - & + \\ & - & \end{matrix} \quad \begin{cases} 0 & -6J-(h-\lambda) \\ + & -4J-2h \end{cases} \qquad \begin{matrix} & 0 & \\ + & - & - \\ & - & \end{matrix} \quad \begin{cases} 0 & -2J-(h-\lambda) \\ + & +4J-2h \end{cases}$$

$$\begin{matrix} & + & \\ + & - & + \\ & + & \end{matrix} \quad \begin{cases} 0 & -12J-(h-\lambda) \\ + & -16J-2h \end{cases} \qquad \begin{matrix} & + & \\ + & - & + \\ & - & \end{matrix} \quad \begin{cases} 0 & -8J-(h-\lambda) \\ + & -8J-2h \end{cases}$$

$$\begin{matrix} & + & \\ + & - & - \\ & - & \end{matrix} \quad \begin{cases} 0 & -4J-(h-\lambda) \\ + & -2h \end{cases} \qquad \begin{matrix} & - & \\ + & - & - \\ & - & \end{matrix} \quad \begin{cases} 0 & -(h-\lambda) \\ + & +8J-2h \end{cases}$$

Fig. 2.

We examine all the possible cases and we show that it is always possible to construct a lower energy configuration by changing the minus spin adjacent to the interface.

Let σ be an acceptable configuration such that there exists only one cluster of 0 spins c^0 and no plus spins; it can be proved that

$$\sigma \text{ is a local minimum of } H(\sigma) \Leftrightarrow \begin{cases} \gamma^0 = \hat{\gamma}^0 \text{ is a rectangle whose} \\ \text{sides are longer than two} \end{cases} \tag{3.1}$$

Indeed, if σ is a local minimum and there exists a minus spin inside the cluster c^0, then, as a consequence of the fact that c^0 does not wind around the torus, one has that necessarily there must exist at least one minus spin with at least two nearest neighbor sites occupied by 0 spins (see Fig. 3). This minus spin can be changed into $+$ or 0 by obtaining, in this way, a lower energy configuration, as shown in Fig. 4; this is absurd.

We can conclude that no minus spins can be inside c^0, that is, $\tilde{\gamma}^0 = \{\varnothing\}$. In a similar way it can be proved that $\hat{\gamma}^0$ is a rectangle and its sides are longer than two.

The proof of the implication $\Leftarrow$ is in Fig. 5 where it is shown that all the possible nearest neighbor configurations of σ are at higher energy; in Fig. 5 the modified spin is represented by a unit empty square.

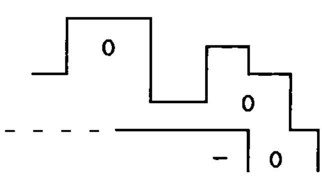


Fig. 3.

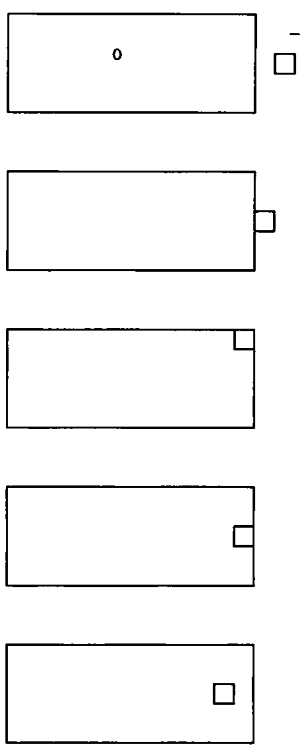
$$\begin{array}{ccc} & 0 & \\ 0 & - & 0 \\ & 0 & \end{array} \quad \begin{cases} 0 & -4J-(h-\lambda) \\ + & -2h \end{cases} \qquad\qquad \begin{array}{ccc} & 0 & \\ - & - & 0 \\ & + & \end{array} \quad \begin{cases} 0 & -(h-\lambda) \\ + & +8J-2h \end{cases}$$

$$\begin{array}{ccc} & 0 & \\ - & - & 0 \\ & 0 & \end{array} \quad \begin{cases} 0 & -2J-(h-\lambda) \\ + & +4J-2h \end{cases}$$

Fig. 4.

Now let σ be an acceptable configuration such that the following conditions are satisfied: there exists just one cluster c^0 of 0 spins touching c^-, $\hat{\gamma}^0$ is a rectangle, no minus spin is inside clusters of 0 spins; all plus spins are in the cluster c^+ and $\hat{\gamma}^+ = \hat{\gamma}^0$ (see Fig. 6). With arguments similar to the ones used before, it can be proved that

$$\sigma \text{ is a local minimum of } H(\sigma) \Leftrightarrow \begin{cases} \gamma^+ = \hat{\gamma}^+ \text{ is a rectangle whose} \\ \text{sides are longer than two} \end{cases} \tag{3.2}$$



$$\begin{cases} 0 & +4J-(h-\lambda) \\ + & +16J-2h \end{cases}$$

$$\begin{cases} 0 & +2J-(h-\lambda) \\ + & +12J-2h \end{cases}$$

$$\begin{cases} - & +(h-\lambda) \\ + & +8J-(h+\lambda) \end{cases}$$

$$\begin{cases} - & +2J+(h-\lambda) \\ + & +6J-(h+\lambda) \end{cases}$$

$$\begin{cases} - & +4J+(h-\lambda) \\ + & +4J-(h+\lambda) \end{cases}$$

Fig. 5.

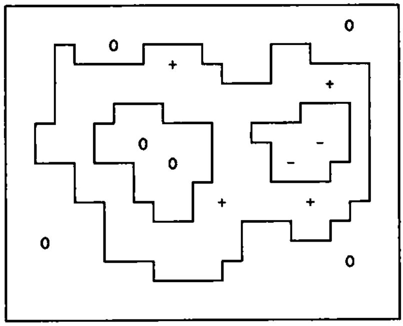


Fig. 6.

In the proof it is crucial that the energy of a configuration can be lowered by properly changing a 0 spin having at most two zero spins and no minus spins among its nearest neighbor sites; all the possible situations are shown in Fig. 7.

Hence we have proved that configurations like the one in Fig. 8 are local minima of $H(\sigma)$; these configurations are called *birectangles* and are denoted by the symbol $R(L_1, L_2; M_1, M_2)$, where

$$\begin{cases} M_1 \geqslant L_1+2, M_2 \geqslant L_2+2 & \text{if} \quad L_1, L_2 \geqslant 2 \\ \qquad M_1, M_2 \geqslant 2 & \text{if} \quad L_1 = L_2 = 0 \end{cases} \tag{3.3}$$

It is easy to understand that the most general local minimum of $H(\sigma)$ is not a birectangle, but, rather, a more complicated configuration that we call family of *plurirectangles* (see Fig. 9). It is an acceptable configuration satisfying the following conditions:

(i) There are k^0 clusters $c_1^0, \ldots, c_{k_0}^0$ of 0 spin touching c^-.

(ii) $\hat{\gamma}_1^0, \ldots, \hat{\gamma}_{k_0}^0$ are noninteracting rectangles whose sides are longer than two.

(iii) In every cluster c_j^0 there are k_j^+ clusters $c_1^+, \ldots, c_{k_j^+}^+$ of $+1$ spins.

(iv) $\forall_j \in \{1, \ldots, k^0\}$, $\hat{\gamma}_{j,1}^+, \ldots, \hat{\gamma}_{j,k_j^+}^+$ are noninteracting rectangles whose sides are longer than two.

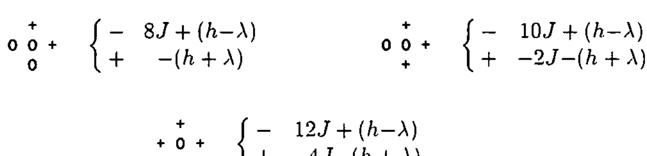


Fig. 7.

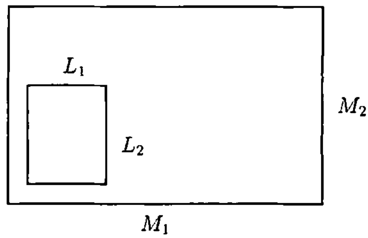


Fig. 8.

We have a single plurirectangle when $k^0 = 1$.

We have used above the geometric notion of interacting rectangles: given two rectangles R_1 and R_2 with boundaries on the dual lattice $\mathbf{Z}^2 + (1/2, 1/2)$, we say that they *interact* if and only if one of the two following conditions occurs:

(i) Their boundaries intersect.

(ii) There exists a unit square centered at some lattice site such that two of its edges are opposite and lie respectively on the boundaries of R_1 and R_2.

We have to compute the energy of such local minima as a first step in the description of the tendency to shrink or grow of the stable clusters.

We say that a local minimum σ is *subcritical* if and only if

$$\lim_{\beta \to \infty} P_\sigma(\tau_{-\underline{1}} < \tau_{+\underline{1}}) = 1 \tag{3.4}$$

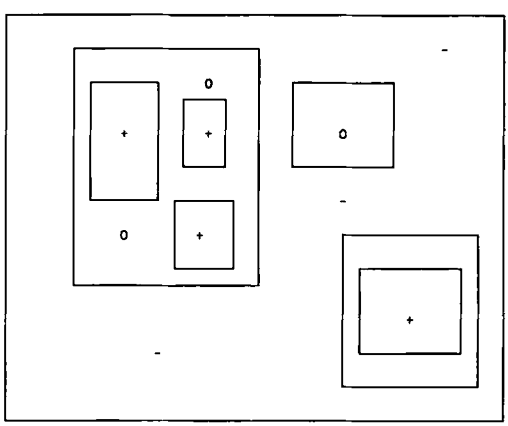


Fig. 9.

One of the main problems that we have to solve is to understand when a local minimum is subcritical.

The energy of a birectangle $R(L_1, L_2; M_1, M_2))$ is

$$H(R(L_1, L_2; M_1, M_2)) - H(-\underline{1})$$
$$= (2M_1 + 2M_2)\, J + (2L_1 + 2L_2)\, J - M_1 M_2(h - \lambda) - L_1 L_2(h + \lambda) \quad (3.5)$$

The above formula can be easily generalized to the case of a general plurirectangle σ, characterized by the parameters $M_{1,j}$, $M_{2,j}$, $L_{1,j,i}$, and $L_{2,j,i}$ $\forall_j \in \{1,..., k^0\}$ and $\forall i \in \{1,..., k_j^+\}$, with obvious meaning of the notation. One has

$$H(\sigma) - H(-\underline{1}) = \sum_{j=1}^{k^0} \{(2M_{1,j} + 2M_{2,j})\, J - M_{1,j} M_{2,j}(h - \lambda)$$
$$+ \sum_{i=1}^{k_j^+} [(2L_{1,j,i} + 2L_{2,j,i})\, J - L_{1,j,i} L_{2,j,i}(h + \lambda)]\} \quad (3.6)$$

Now we consider a squared birectangle $Q(L, M) := R(L, L; M, M)$, whose energy $e(M, L) := H(Q(L, M)) - H(-\underline{1})$ is given by

$$e(M, L) = 4MJ + 4LJ - M^2(h - \lambda) - L^2(h + \lambda) \quad (3.7)$$

The graph of this function $e: \mathbf{R}^2 \to \mathbf{R}$ is a paraboloid with elliptical section and downhill concavity; the coordinates of the vertex are

$$M = \frac{2J}{h - \lambda}, \qquad L = \frac{2J}{h + \lambda} \quad (3.8)$$

Let us consider a droplet $Q(M, L)$ such that $M < 2J/(h - \lambda)$ and $L < 2J/(h + \lambda)$: if these conditions are satisfied, $e(M, L)$ is an increasing function of M and L, so we expect that this droplet will shrink. On the other hand, if $M > 2J/(h - \lambda)$, since $e(M, L)$ is a decreasing function of M, we expect that the external cluster of the droplet will grow; this suggests that $M^* := [2J/(h - \lambda)] + 1$ is the *critical dimension* for the external cluster of a local minimum. After the growth of the external cluster, we look at what will happen to the internal one; with similar arguments one can convince oneself that $L^* := [2J/(h + \lambda)] + 1$ appears to play the role of the critical dimension. Obviously these two processes of growth cannot be inverted; in fact, a plus spin droplet can "live" only inside a zero spin droplet.

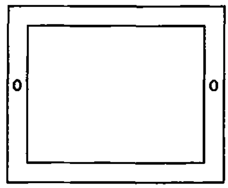


Fig. 10.

But it can also happen that the plus spin phase is reached directly, without passing through the zero spin phase; this happens if the droplet $Q(M, L)$ grows moving along the line $M = L+2$. In this case one can see that the system reaches the stable phase through a sequence of frames (picture frames). We call a *squared frame* a birectangle $C(l, l) := R(l, l; l+2, l+2)$ with $l \geqslant 2$. The most general *frame* is a rectangular one (see Fig. 10):

$$C(l_1, l_2) := R(l_1, l_2; l_1+2, l_2+2) \tag{3.9}$$

where $l_1, l_2 \geqslant 2$.

Now we consider the energy of a squared frame $e(l) := H(C(l, l)) - H(-\underline{1})$; using equality (3.7), we have

$$e(l) = -2hl^2 + l[8J - 4(h-\lambda)] + [8J - 4(h-\lambda)] \tag{3.10}$$

The graph of this function is a concave parabola, whose vertex coordinate is

$$l = \frac{2J - (h-\lambda)}{h} \tag{3.11}$$

We expect that $C(l, l)$ will grow if $l \geqslant l^*$, where

$$l^* := \left[\frac{2J - (h-\lambda)}{h}\right] + 1 \tag{3.12}$$

Otherwise it will shrink; hence l^* should be the critical dimension of a squared frame.

In order to describe the behavior of a general birectangle $R = R(L_1, L_2; M_1, M_2)$, we must study the growth and contraction mechanisms of a droplet; as in the Ising model, these are mainly growth of a (unit square) protuberance and corner erosion. But in the Blume–Capel

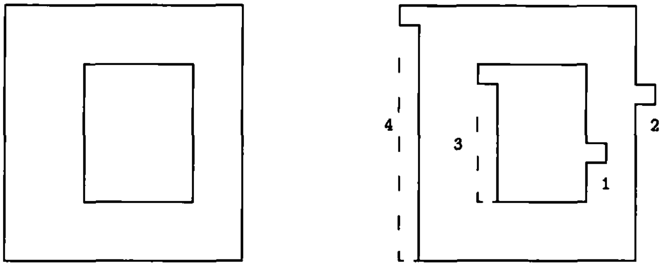


Fig. 11.

model the relevant local minima have two components, internal and external ones, and they can grow or shrink independently. The mechanisms of growth and contraction are explained in Fig. 11, and correspond to:

1. Creation of a + protuberance adjacent from the exterior to the internal rectangle.
2. Creation of a 0 protuberance adjacent from the exterior to the external rectangle.
3. Erosion ($+ \to 0$) of all but one + spin in a row or column of the internal rectangle.
4. Erosion ($0 \to -$) of all but one 0 spin in a row or column of the external rectangle.

Their typical times are

$$t_1 = e^{\beta[2J-(h+\lambda)]}, \qquad t_2 = e^{\beta[2J-(h-\lambda)]}$$
$$t_3 = e^{\beta(h+\lambda)(L-1)}, \qquad t_4 = e^{\beta(h-\lambda)(M-1)} \tag{3.13}$$

where $L := \min\{L_1, L_2\}$ and $M := \min\{M_1, M_2\}$.

By comparing times $t_1,..., t_4$, we observe that the growth of an internal protuberance is always faster than the growth of an external one, indeed

$$2J-(h+\lambda) < 2J-(h-\lambda) \Rightarrow t_1 < t_2 \tag{3.14}$$

Then we introduce the following critical dimensions:

$$L^* := \left[\frac{2J}{h+\lambda}\right]+1, \qquad \tilde{L} := \left[\frac{2J+2\lambda}{h+\lambda}\right]+1$$
$$M^* := \left[\frac{2J}{h-\lambda}\right]+1, \qquad \tilde{M} := \left[\frac{2J-2\lambda}{h-\lambda}\right]+1 \tag{3.15}$$

with the following meaning:

- $L < L^* \Leftrightarrow (h+\lambda)(L-1) < 2J-(h+\lambda)$: internal contraction is faster than growth, that is, the internal component of the local minimum is (relatively) *subcritical.*
- $L < \tilde{L} \Leftrightarrow (h+\lambda)(L-1) < 2J-(h-\lambda)$: internal contraction is faster than external growth.
- $M < \tilde{M} \Leftrightarrow (h-\lambda)(M-1) < 2J-(h+\lambda)$: external contraction is faster than internal growth.
- $M < M^* \Leftrightarrow (h-\lambda)(M-1) < 2J-(h-\lambda)$: external contraction is faster than growth, that is, the external component of the local minimum is (relatively) *subcritical.*

As we will see in the next section, another interesting length will be $l_0 := [h/\lambda]+1$.

We choose the parameters J, h, and λ in such a way that $2J/(h+\lambda)$, $(2J+2\lambda)/(h+\lambda)$, $2J/(h-\lambda)$, $(2J-2\lambda)/(h-\lambda)$, $(2J-(h-\lambda))/h$, and h/λ are not integer, so that ambiguous situation here and in the following, are avoided.

The behavior of our birectangle R depends on its dimensions; some of the possible cases are as follows:

- $L < L^*$ and $M < M^*$: both internal and external component are subcritical; R is subcritical.
- $L < L^*$ and $M > M^*$: the internal component is subcritical but not the external one, R is supercritical and the system starting from R will reach $\underline{0}$.
- $L > L^*$ and $M > M^*$: both internal and external components are supercritical; R is supercritical and the system starting from R will reach $+\underline{1}$ by passing through $C(M_1-2, M_2-2)$ (internal growth is faster than external growth).
- $L > L^*$ and $M < M^*$: the internal component is supercritical, while the external one is subcritical; the future of the system starting from R depends on the relation $M \gtrless \tilde{M}$.

Many different situations can take place; the last one is surely the most difficult, but also the most interesting that we have to examine.

Growth and contraction of a frame are based on the same elementary mechanisms described before, but they take place in more than one step. The possible contraction of a squared frame $C(l, l)$ starts with the contraction of its internal component: our system typically first reaches the

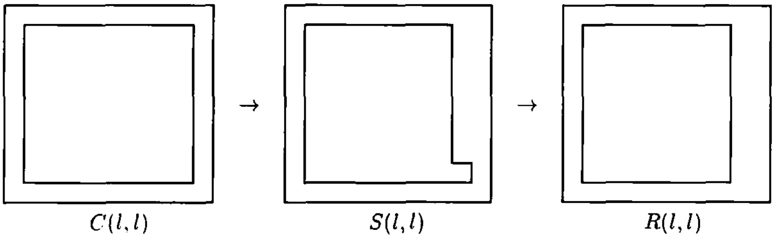


Fig. 12.

configuration $S(l, l)$, increasing the energy of the quantity $H(S(l, l)) - H(C(l, l)) = (h + \lambda)(l - 1)$, and then the configuration $R(l, l) := R(l - 1, l; l + 2, l + 2)$, lowering the energy of the quantity $H(S(l, l)) - H(R(l, l)) = 2J - (h + \lambda)$ (see Fig. 12).

At this level it is not easy to describe the future evolution of the system: the internal component could continue to shrink or the external component could start its contraction; but we remark that the first step in the contraction of $C(l, l)$ always involves bypassing of an energetic barrier whose height is $(h + \lambda)(l - 1)$.

On the other hand, the possible expansion of $C(l, l)$ starts with the growth of an external protuberance: the system typically reaches the configuration $G(l, l)$ by overcoming the energetic barrier $H(G(l, l)) - H(C(l, l)) = 2J - (h - \lambda)$ and then it goes down to $R(l + 1, l) := R(l, l; l + 3, l + 2)$ lowering the energy of the quantity $H(G(l, l)) - H(R(l + 1, l)) = (h - \lambda)(l + 1)$ (see Fig. 13). We have supposed, without loss of generality, that the growth is horizontal.

As a consequence of the fact that one always has $t_1 < t_2$, the second step in the expansion of the droplet will be the growth of an internal protuberance: the system reaches the configuration $S(l + 1, l)$ by overcoming the energetic barrier $H(S(l + 1, l)) - H(R(l + 1, l)) = 2J - (h + \lambda)$ and then goes down to the frame $C(l + 1, l)$, lowering the energy of the quantity $H(S(l + 1, l)) - H(C(l + 1, l)) = (h + \lambda)(l - 1)$.

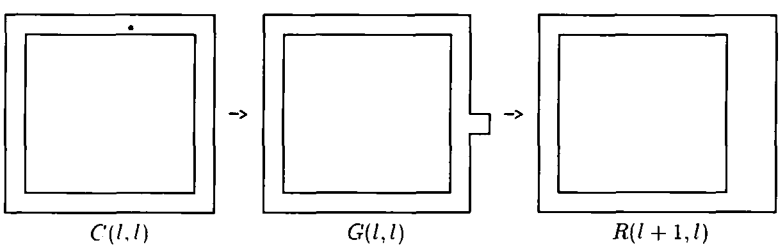


Fig. 13.

In order to describe the future probable evolution of the system starting from $C(l, l)$, and establish its tendency to shrink or grow, we have to distinguish the following four cases:

$$l < \tilde{L} \Rightarrow H(S(l, l)) < H(G(l, l))$$

$$\tilde{L} < l < l^* \Rightarrow \begin{cases} H(G(l, l)) < H(S(l, l)) \\ H(S(l, l)) < H(S(l+1, l)) \end{cases}$$

$$l^* < l,\ l+2 < \tilde{M} \Rightarrow \begin{cases} H(G(l, l)) < H(S(l+1, l)) \\ H(S(l+1, l)) < H(S(l, l)) \end{cases}$$

$$l^* < l,\ \tilde{M} < l+2 \Rightarrow \begin{cases} H(G(l, l)) < H(S(l, l)) \\ H(S(l+1, l)) - H(R(l+1, l)) \\ \qquad < H(G(l, l)) - H(R(l+1, l)) \end{cases}$$

These four cases are illustrated in Fig. 14.

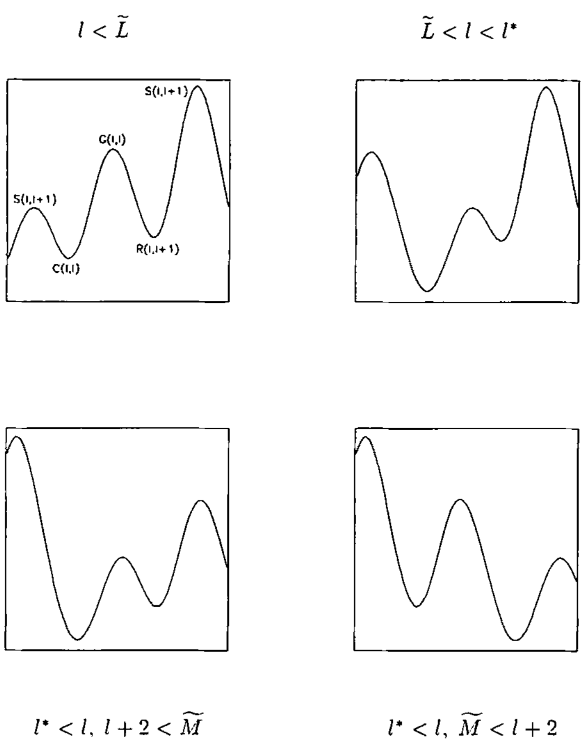


Fig. 14.

The analysis of the growth and contraction mechanisms leads to the conclusion that l^* is the critical dimension of a square frame.

We close this section by remarking that the parameter λ may be chosen sufficiently small, that is,

$$\lambda < \frac{2J}{2a^2 + a - 1} \tag{3.16}$$

where $a = b/\lambda$, so that the following inequalities are satisfied:

$$L^* + 1 \leqslant l^* \tag{3.17a}$$

$$L^* \leqslant \tilde{L} < l^* < l^* + 3 \leqslant \tilde{M} \leqslant M^* \tag{3.17b}$$

4. SUPERCRITICALITY AND SUBCRITICALITY OF LOCAL MINIMA

In this section we want to prove rigorous results about supercriticality and subcriticality of local minima. Namely, we want to give criteria to establish the natural tendency of the geometrical structures representing the minima for H to shrink or grow. We will first analyze the "frames," then the generic birectangles, and finally the plurirectangles.

First we state the following proposition:

Proposition 4.1. Let us consider the configuration $C(l_1, l_2)$; we set $l := \min\{l_1, l_2\}$ and $m := \max\{l_1, l_2\}$. Let $\varepsilon > 0$; we have

$$l < l^* \text{ and } m < m^*(l) \Rightarrow \begin{cases} \lim_{\beta \to \infty} P_{C(l_1, l_2)}(\tau_{-\underline{1}} < \tau_{+\underline{1}}) = 1 \\ \lim_{\beta \to \infty} P_{C(l_1, l_2)}(T^s_- < \tau_{-\underline{1}} < T^s_+(\varepsilon)) = 1 \end{cases}$$

where

$$T^s_{\pm}(\varepsilon) := \begin{cases} e^{\beta(h-\lambda)(l+1) \pm \beta\varepsilon} & l < \left[\frac{h}{\lambda}\right] + 1 \\ e^{\beta(h+\lambda)(l-1) \pm \beta\varepsilon} & l \geqslant \left[\frac{h}{\lambda}\right] + 1 \end{cases}$$

and

$$m^*(l) := \left[\frac{2h}{h-\lambda}\,\frac{2J-(h-\lambda)}{h} - \frac{h+\lambda}{h-\lambda}\,l\right] + 1$$

Moreover,

$$\tilde{L} \leqslant l < l^* \text{ and } m \geqslant m^*(l) \Rightarrow \begin{cases} \lim_{\beta \to \infty} P_{C(l_1,l_2)}(\tau_{+\underline{1}} < \tau_{-\underline{1}}) = 1 \\ \lim_{\beta \to \infty} P_{C(l_1,l_2)}(T_-^{g,1}(\varepsilon) < \tau_{+\underline{1}} < T_+^{g,1}(\varepsilon)) = 1 \end{cases}$$

where

$$T_\pm^{g,1}(\varepsilon) := \begin{cases} e^{\beta\{[2J-(h-\lambda)]-(h-\lambda)(m+1)+[2J-(h+\lambda)]\} \pm \beta\varepsilon} & m < \tilde{M} - 2 \\ e^{\beta[2J-(h-\lambda)] \pm \beta\varepsilon} & m \geqslant \tilde{M} - 2 \end{cases}$$

Finally

$$l \geqslant l^* \Rightarrow \begin{cases} \lim_{\beta \to \infty} P_{C(l_1,l_2)}(\tau_{+\underline{1}} < \tau_{-\underline{1}}) = 1 \\ \lim_{\beta \to \infty} P_{C(l_1,l_2)}(T_-^{g,2}(\varepsilon) < \tau_{+\underline{1}} < T_+^{g,2}(\varepsilon)) = 1 \end{cases}$$

where

$$T_\pm^{g,2}(\varepsilon) := \begin{cases} e^{\beta\{[2J-(h-\lambda)]-(h-\lambda)(m+1)+[2J-(h+\lambda)]\} \pm \beta\varepsilon} & l, m < \tilde{M} - 2 \\ e^{\beta[2J-(h-\lambda)] \pm \beta\varepsilon} & \text{otherwise} \end{cases}$$

Proof. Let us consider the frame $C := C(l_1, l_2)$ with $l := \min\{l_1, l_2\} < \tilde{L}$, its basin of attraction $B := B(C(l_1, l_2))$, and the relative boundary ∂B. Let us denote by $S_1 \in \partial B$ the set of configurations obtained by changing into zero $l-1$ plus spins adjacent to one of the shortest sides of the internal rectangle of C (see Fig. 15); we claim that

$$\min_{\sigma \in \partial B} H(\sigma) = H(S_1) \tag{4.1}$$

Remark. In the following we will consider:

1. Configurations σ containing a unique droplet γ with a given particular shape, size, and location; for example, a rectangle of zeros (with given location and horizontal and vertical sizes) or a birectangle with given location and external and internal horizontal and vertical sizes.

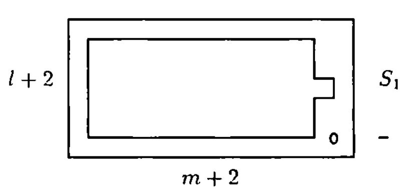
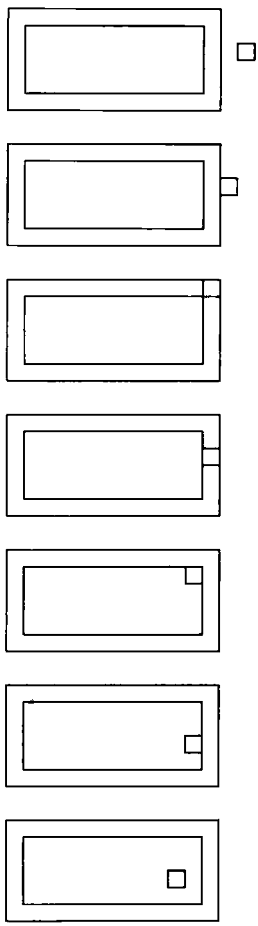


Fig. 15.

2. The equivalence class of all the configurations σ' with a unique droplet γ' obtained from γ by symmetries such as rotations, translations, inversions w.r.t lattice axes, and even displacements along sides of unit-square protuberances.

In the following, to avoid lengthy specification and to accelerate the exposition, we often interchange the above two objects and we even use the same symbols to denote them. The reader will easily deduce the meaning of our statements from the context.

For example, sometimes we will denote by S_1 also a particular droplet obtained from a particular configuration in C by substituting one particular smaller internal side with a particular unit-square protuberance.



$$\begin{cases} + & \Delta H_{11} = 16J-2h \\ 0 & \Delta H_{12} = 4J-(h-\lambda) \end{cases}$$

$$\begin{cases} + & \Delta H_{21} = 12J-2h \\ 0 & \Delta H_{22} = 2J-(h-\lambda) \end{cases}$$

$$\begin{cases} + & \Delta H_{31} = 8J-(h+\lambda) \\ - & \Delta H_{32} = (h-\lambda) \end{cases}$$

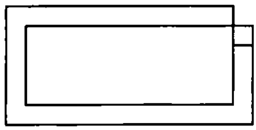
$$\begin{cases} + & \Delta H_{41} = 4J-(h+\lambda) \\ - & \Delta H_{42} = 4J+(h-\lambda) \end{cases}$$

$$\begin{cases} 0 & \Delta H_{51} = +(h+\lambda) \\ - & \Delta H_{52} = 8J+2h \end{cases}$$

$$\begin{cases} 0 & \Delta H_{61} = 2J+(h+\lambda) \\ - & \Delta H_{62} = 12J+2h \end{cases}$$

$$\begin{cases} 0 & \Delta H_{71} = 4J+(h+\lambda) \\ - & \Delta H_{72} = 16J+2h \end{cases}$$

Fig. 16.



$$\begin{cases} + & \Delta H_{81} = 6J-(h+\lambda) \\ - & \Delta H_{82} = 2J+(h-\lambda) \end{cases}$$

Fig. 17.

Let us now continue the proof of Proposition 4.1.

In order to prove (4.1), we observe that, starting from C and considering all the possible uphill paths, one is able to examine all the configurations in ∂B. The energy costs of all the possible first steps of the above-mentioned paths are given in Fig. 16; here we mark by a unitary square the site whose spin is changed and we denote by a pair of positive integer numbers (i, j) the generic first step of our uphill path. We denote by $C_{i,j}$ the configuration reached after the step (i, j). We observe that $C_{2,2} \in \partial B$ and that $\Delta H_{ij} > \Delta H_{22} \forall (i,j) \notin \{(5,1),(3,2),(2,2)\}$. Hence, all the paths whose first step is different from (5, 1) and (3, 2) lead to a boundary configuration whose energy is greater than $H(C_{2,2})$.

Starting from $C_{3,2}$ or $C_{5,1}$ an uphill path can continue by following one of the ways shown in Fig. 16 and 17. It can be easily shown that the steps $(8, j)$ can be neglected as well.

In conclusion, only the paths made by steps (3, 2) and (5, 1) can lead to a configuration whose energy is lower than $H(C_{2,2})$.

Now, let σ be an acceptable configuration such that the following conditions are satisfied: there exists just one cluster c^0 of 0 spins which touches the sea of minuses, namely the cluster c^- winding around the torus; no minus spins are inside c^0; all plus spins are in a unique cluster c^+ included in c^0 and $\hat{\gamma}^+ = \hat{\gamma}^0$. If $\sigma \in B$, then the following propositions are true:

(i) $R(c^0) \equiv$ the external rectangle $(l_1+2) \times (l_2+2)$ of the frame C.

(ii) $R(c^+) \equiv$ the internal rectangle $l_1 \times l_2$ of the frame C.

(iii) The intersection of each one of the four sides of $R(c^0)$ with $\hat{\gamma}^0$ contains at least a segment of length greater than or equal to 2.

(iv) The intersection of each one of the four sides of $R(c^+)$ with $\hat{\gamma}^+$ contains at least a segment of length greater than or equal to 2.

We prove (i) by absurdity: let us suppose that $R(c^0)$ is different from $(l_1+2) \times (l_2+2)$ and that $\hat{\gamma}^+$ is a rectangle. We can construct a downhill path which leads to a local minimum different from C by filling with 0 spins the region $R(c^0) \backslash c^+$. Thus $\sigma \notin B$, and this is absurd. Statement (ii) can be proved in a similar way. Statement (iii) is proved by absurdity as well: suppose that the intersection between $\hat{\gamma}^0$ and one of the sides of $R(c^0)$

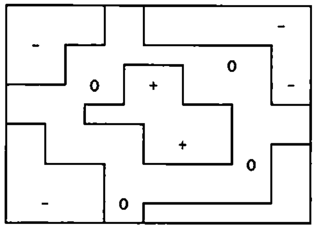


Fig. 18.

contains only isolated intervals of length 1; namely, there is a certain number of spins 0 with three minus spins among their nearest neighbor sites. By changing these 0 spins into -1, we construct a configuration at a lower energy level and characterized by a cluster of 0 spins c'^0 such that $R(c'^0)$ is different from the rectangle $(l_1+2)\times(l_2+2)$; then there exists a downhill path which connects σ to a local minimum different from C. Hence the absurdity $\sigma \notin B$ is obtained. Statement (iv) is proved in a similar way.

But, as we noticed before, all the uphill paths starting from C and leading to configurations in ∂B with energy smaller than $H(C_{22})$ necessarily can only be made by steps (5, 1) and (3, 2).

It is clear that, by virtue of the necessary conditions stated above, we cannot reach ∂B starting from C with fewer than $l-1$ steps (5, 1). On the other hand, since $S_1 \in \partial B$, with more than $l-1$ steps (5, 1) we certainly get an energy larger that $H(S_1)$ and so a configuration which cannot be of minimal energy in ∂B.

In this way we can only reach configurations with a unique cluster of pluses, so any boundary configuration with minimal energy is characterized by an external cluster c^0, such that the intersection between $\hat{\gamma}^0$ and all the sides of $R(c^0)$ is at least of length 2, and an internal cluster c^+, such that the intersection between $\hat{\gamma}^+$ and one of the sides of $R(c^+)$ has length 1 (see Fig. 18). Among all these configurations it is easily seen that the one with lowest energy is S_1.

In conclusion we have to compare $H(S_1)$ with $H(C_{2,2})$. Equality (4.1) follows from $l<\tilde{L}$, $H(S_1)-H(C)=(h+\lambda)(l-1)$, and $H(C_{22})-H(C)=2J-(h-\lambda)$.

Now we want to apply to the description of the first escape from B the approach developed in ref. 15, which is based on the properties of the above-defined sets called cycles.

It is easy to see that the basin of attraction $B:=B(C(l_1,l_2))$ defined in (2.13) satisfies the following properties:

(i) B is connected.

(ii) $S_1 \subset \partial B$, and

$$\min_{\sigma \in \partial B} H(\sigma) = H(S_1), \qquad \min_{\sigma \in \partial B \setminus S_1} H(\sigma) > H(S_1)$$

(iii) $\forall \eta \in S_1$ there exists a path $\omega: \eta \to C$ such that $\forall \sigma \in \omega \setminus \{\eta\}$ one has $\sigma \in B$ and $H(\sigma) < H(S_1)$.

As was noticed in ref. 15 (see Proposition 3.4 therein), properties (i)–(iii) imply that the set $\bar{B}$ defined as the maximal connected set containing C and with energy less than $H(S_1)$ is a cycle with S_1 belonging to its boundary $\partial \bar{B}$. Moreover, we notice the following obvious properties:

- Any point $\eta \in \partial \bar{B}$ necessarily is such that $H(\eta) \geqslant H(S_1)$.
- If $H(\eta) = H(S_1)$ and $\eta \notin \partial B$, necessarily any downhill path starting from η ends in C.

We recall that, given any set $A \subset \Omega_\Lambda$, we have denoted by $\mathscr{S}(A)$ the possibly empty subset of $U(A)$ [see (2.10)], which is downhill connected to A^c; $\mathscr{S}(A)$ was called the set of minimal saddles of A. We can write

$$\mathscr{S}(\bar{B}) = S_1 \tag{4.2}$$

From Proposition 3.7 in ref. 15, from reversibility of the dynamics (see Lemma 1 in ref. 9), and from (4.2) we easily get that $\forall \sigma \in \bar{B}$

$$\lim_{\beta \to \infty} P_\sigma(\sigma_{\tau_{(B \cup \partial B)^c} - 1} \in S_1) = 1 \tag{4.3}$$

Since $H(S_1) - H(C) = (h + \lambda)(l - 1)$, we deduce that for every $\varepsilon > 0$

$$\lim_{\beta \to \infty} P_C(e^{\beta(h+\lambda)(l-1) - \beta\varepsilon} < \tau_{\partial B} < e^{\beta(h+\lambda)(l-1) + \beta\varepsilon}) = 1 \tag{4.4}$$

Up to now we have described how the system reaches ∂B starting from C; now we want to describe its further evolution.

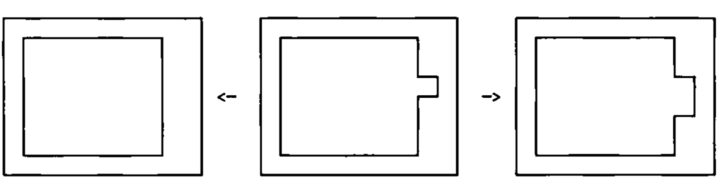


Fig. 19.

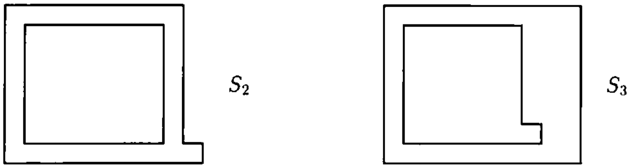


Fig. 20.

Two things can happen: the system gets back to B or it goes to the birectangle $R_1 := R(l_1 - 1, l; l_1 + 2, l+2)$ (see Fig. 19); we have supposed, without loss of generality, that $l = l_2$.

In Appendix A we give a general argument showing that, with high probability, our process, possibly after many attempts, sooner or later will eventually get out of $B \cup \partial B$ through S_1 reaching R_1 before touching any other local minimum and

$$\lim_{\beta \to \infty} P_C(\tau_{R_1} < \tau_{+1}) = 1$$
$$\lim_{\beta \to \infty} P_C(\tau_{R_1} < e^{\beta(h+\lambda)(l-1)+\beta\varepsilon}) = 1 \tag{4.5}$$

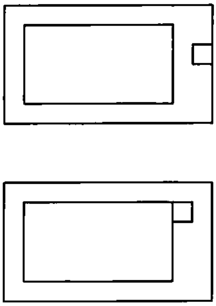
Now we have to describe the further evolution of our Markov chain starting from the birectangle R_1. We denote by $B_1 := B(R_1)$ the basin of attraction of R_1. Let us first consider the case $\min\{l_1 - 1, l\} = l$ (this is equivalent to supposing that C is not a squared frame). We denote by S_2 the configuration obtained by changing into minus $l+1$ of the 0 spins on the "free" side of the external rectangle and by S_3 the configuration obtained by changing into zero $l-1$ of the plus spins of one of the shortest sides of the internal rectangle of R_1 (see Fig. 20). The following is true:

$$\min_{\sigma \in \partial B_1} H(\sigma) = \begin{cases} H(S_3) & \text{if } l < \left[\dfrac{h}{\lambda}\right] + 1 \\ H(S_2) & \text{if } l \geqslant \left[\dfrac{h}{\lambda}\right] + 1 \end{cases} \tag{46}$$

Equality (4.6) can be proved with arguments similar to those used in the case of the local minimum $C(l_1, l_2)$ and observing that $H(S_2) - H(R_1) = (h-\lambda)(l+1)$ and $H(S_3) - H(R_1) = (h+\lambda)(l-1)$ even though, in this case, there are other possible first steps (with high increment in energy). They are shown in Fig. 21.

With arguments similar to those used before, we get that the typical time of first escape from ∂B_1 is of the order of

$$\exp\{\beta[\min_{\sigma \in \partial B_1} H(\sigma) - H(R_1)]\}$$



$$\begin{cases} + & \Delta H_{91} = 6J-(h+\lambda) \\ - & \Delta H_{82} = 2J+(h-\lambda) \end{cases}$$

$$\begin{cases} + & \Delta H_{10\,1} = 2J-(h+\lambda) \\ - & \Delta H_{10\,2} = 6J+(h-\lambda) \end{cases}$$

Fig. 21.

and that the system hits for the first time the boundary ∂B_1 in S_3 if $l < [h/\lambda]+1$ and in S_2 if $l \geqslant [h/\lambda]+1$. Notice that if $1 < h/\lambda < 2$, the integer $[h/\lambda]+1$ equals 2, so that S_2 is preferred.

We have that, with probability tending to 1 as $\beta \to \infty$, our droplet continues its contraction: the system reaches another local minimum R_2 strictly contained in R_1, that is

$$R_2 \prec R_1 \prec C \tag{4.7}$$

where we have introduced the following *partial order relation* in Ω_Λ:

$$\sigma \prec \eta \Leftrightarrow \sigma(x) \leqslant \eta(x) \quad \forall x \in \Lambda \tag{4.8}$$

We also have that, given $\varepsilon > 0$,

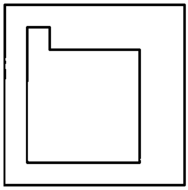
$$\exp\{\beta[\min_{\sigma \in \partial B_1} H(\sigma) - H(R_1)] + \beta\varepsilon\}$$

is an upper bound, in the limit $\beta \to \infty$, to the first hitting time to R_2 of the Markov chain starting from R_1.

In conclusion, we can say that the Markov chain starting from C visits smaller and smaller local minima until it reaches the configuration $-\underline{1}$; this completes the proof of the statement $P_C(\tau_{-\underline{1}} < \tau_{+\underline{1}}) \xrightarrow{\beta \to \infty} 1$.

Each step of the shrinking process is characterized by a typical time t_β whose asymptotic behavior, exponentially in β, is known in the sense that we control

$$\lim_{\beta \to \infty} \frac{1}{\beta} \log t_\beta$$



$l+2$

$l+2$

Fig. 22.

We say that

$$t^1_\beta,\ t^2_\beta \textit{ are logarithmically equivalent}$$

$$\Leftrightarrow \lim_{\beta\to\infty}\frac{1}{\beta}\log t^1_\beta=\lim_{\beta\to\infty}\frac{1}{\beta}\log t^2_\beta$$

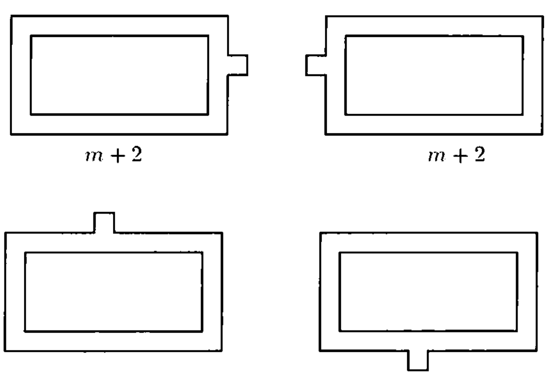
By using Markov property, we can say that the typical time of the whole shrinking event is given by the largest time among all the *partial shrinking times*. Then the proof of Proposition 4.1 is completed in the case $l<\tilde{L}$ when C is a rectangular frame.

Next, we consider the case when C is a squared frame: the boundary configuration S_3 is now the one represented in Fig. 22; $H(S_3)-H(R_1)=(h+\lambda)(l-2)$ and $\min_{\sigma\in\partial B_1}H(\sigma)=H(S_3)$ if $l<[\frac{3}{2}h/\lambda+\frac{1}{2}]+1$. We obtain results similar to those obtained in the previous case of a rectangular frame.

Now we consider the frame $C:=C(l_1,l_2)$; we suppose that $\tilde{L}<l:=\min\{l_1,l_2\}<l^*$ and $m:=\max\{l_1,l_2\}<m^*(l)$; we denote by B the basin of attraction of the frame C and by ∂B its boundary. We denote by S_4 the

$S_{4,\perp}$

$S_{4,\parallel}$



$l+2$

$m+2$ $m+2$

$l+2$

$m+2$ $m+2$

Fig. 23.

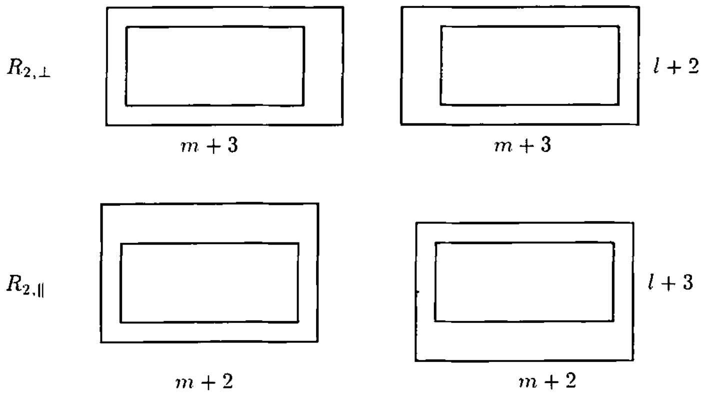


Fig. 24.

set of configurations obtained by attaching a unit-square protuberance (with a zero spin) to one of the four side's of the external rectangle of C (see Fig. 23). By considering all the uphill paths starting from C, it can be proved that

$$\min_{\sigma \in \partial B} H(\sigma) = H(S_4), \qquad \min_{\sigma \in \partial B \setminus S_4} H(\sigma) < H(S_4) \tag{4.9}$$

namely

$$U(B) = S_4 \tag{4.10}$$

We remark that $H(S_4) - H(C) = 2J - (h - \lambda)$.

Without loss of generality we suppose that $l = l_2$; and $m = l_1$. By arguments similar to those used before, it can be proved that in a typical time $e^{[2J-(h-\lambda)]}$ the Markov chain starting from C, with high probability, will visit $R_{2,\perp} := R(l_1, l_2; l_1+3, l_2+2)$ or $R_{2,\parallel} := R(l_1, l_2; l_1+2, l_2+3)$. The symbol $\perp$ denotes the fact that the frame is growing in a direction perpendicular to its shortest side (see Fig. 24).

We denote by $B_{2,\perp}$ and $B_{2,\parallel}$ the basins of attraction of $R_{2,\perp}$ and $R_{2,\parallel}$; their boundaries are respectively denoted by $\partial B_{2,\perp}$ and $\partial B_{2,\parallel}$. By considering all the uphill paths starting from $R_{2,\perp}$ and $R_{2,\parallel}$, we get that

$$\begin{aligned} \min_{\sigma \in \partial B_{2,\perp}} H(\sigma) &= H(S_4) \\ \min_{\sigma \in \partial B_{2,\parallel}} H(\sigma) &= H(S_4) \end{aligned} \tag{4.11}$$

More precisely,

$$U(B_{2,\perp}) = S_4, \qquad U(B_{2,\parallel}) = S_4 \tag{4.12}$$

Indeed the most relevant inequalities in the proof of (4.11) are

$$\begin{gathered}(h+\lambda)(l-1) > 2J-(h-\lambda) > 2J-(h+\lambda) > (h-\lambda)(l+1)\\ (h+\lambda)(l-1) > 2J-(h-\lambda) > 2J-(h+\lambda) > (h-\lambda)(m+1)\end{gathered} \tag{4.13}$$

We remark that $H(S_4) - H(R_{2,\perp}) = (h-\lambda)(l+1)$ and $H(S_4) - H(R_{2,\parallel}) = (h-\lambda)(m+1)$. In order to prove the first of the equalities (4.13), we notice that

$$l \geq \tilde{L} \Rightarrow (h+\lambda)(l-1) > 2J-(h-\lambda)$$

$$l+2 < \tilde{M} \Rightarrow 2J-(h+\lambda) > (h-\lambda)(l+1)$$

In order to prove the second one, we notice that

$$l \geq \tilde{L} \Rightarrow m^*(l) + 2 \leq \tilde{M}$$

and that

$$m < m^*(l) \Rightarrow m+2 < \tilde{M} \Rightarrow (h-\lambda)(m+1) < 2J-(h+\lambda)$$

Starting from $R_{2,\perp}$ or from $R_{2,\parallel}$ the system will typically go back to C before visiting other frames; these phenomena take place, respectively, in the two typical times $e^{\beta(h-\lambda)(l+1)}$ and $e^{\beta(h-\lambda)(m+1)}$. It appears clear that the system, before eventually leaving C to reach another frame, will wander, performing random oscillations, in the union of the basins B, $B_{2,\perp}$, and $B_{2,\parallel}$. Then, in order to understand whether the frame will shrink or grow we have to describe its behavior in a larger basin, containing $B \cup B_{2,\perp} \cup B_{2,\parallel}$. This basin is denoted by $\mathscr{D}$ and it is defined as follows

$$\mathscr{D} := \{\eta\text{: every downhill path starting from } \eta \text{ ends in } C \text{ or } R_{2,\perp} \text{ or } R_{2,\parallel}\} \tag{4.14}$$

We denote by $S_{5,\perp}$ and $S_{5,\parallel}$ the configurations obtained by attaching a unit-square protuberance to the free side of the internal rectangle of $R_{2,\perp}$ and $R_{2,\parallel}$ (see Fig. 25). By considering all the uphill paths starting from C, $R_{2,\perp}$, and $R_{2,\parallel}$, we are able to examine all the configurations in $\partial\mathscr{D}$. We get

$$\min_{\sigma \in \partial\mathscr{D}} H(\sigma) = H(S_1), \qquad U(\mathscr{D}) = S_1 \tag{4.15}$$

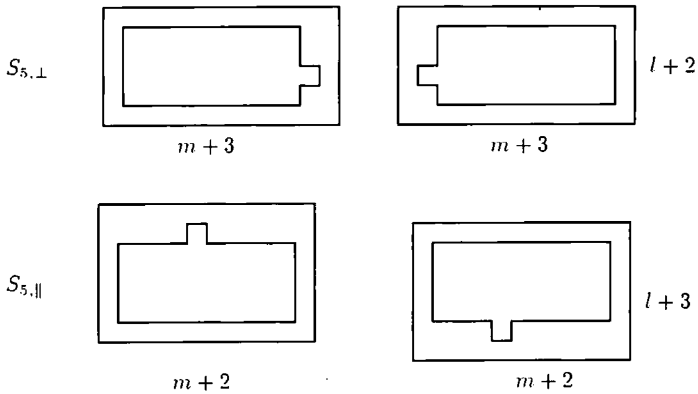


Fig. 25.

The most relevant inequalities in the proof of Eq. (4.15) are

$$l<l^* \Rightarrow (h+\lambda)(l-1) < [2J-(h-\lambda)] - (h-\lambda)(l+1) + [2J-(h+\lambda)]$$

$$m<m^*(l) \Rightarrow (h+\lambda)(l-1) < [2J-(h-\lambda)] - (h-\lambda)(m+1) + [2J-(h+\lambda)] \tag{4.16}$$

They mean, respectively, $H(S_{5,\perp}) > H(S_1)$ and $H(S_{5,\parallel}) > H(S_1)$. Of course one always has $H(S_{5,\parallel}) < H(S_{5,\perp})$.

We notice that $\mathscr{D}$ is a sort of generalized basin of attraction of C; indeed it is easy to see that as a consequence of $m<M^*$ the "bottom" of $\mathscr{D}$ reduces to C in the sense that C is the only absolute minimum of the energy in $\mathscr{D}$ and, as it is easy to see, starting from any initial configuration $\sigma \in \mathscr{D}$ our process, with high probability for large β, will visit C before exiting from $\mathscr{D}$. From $\mathscr{D}$ one can easily obtain, by suitably cutting in energy, a cycle having the same minimal saddles in its boundary: take the maximal connected set $\bar{\mathscr{D}}$ of configurations containing C with energy less than $H(S_1)$. Since it is easy to see that properties (i)–(iii) of $B=B(C(l_1, l_2))$ still hold with $\mathscr{D}$ in place of B for $\tilde{L}<l<l^*$, $m<m^*(l)$, one immediately gets $\mathscr{S}(\bar{\mathscr{D}}) \ni S_1$; moreover, $\forall \sigma \in \mathscr{D}$,

$$\lim_{\beta \to \infty} P_\sigma(\sigma_{\tau_{(\mathscr{D}\cup\partial\mathscr{D})^c}-1} \in S_1) = 1$$

and for every $\varepsilon>0$

$$\lim_{\beta \to \infty} P_C(e^{\beta(h+\lambda)(l-1)-\beta\varepsilon} < \tau_{\partial\mathscr{D}} < e^{\beta(h+\lambda)(l-1)+\beta\varepsilon}) = 1$$

We want to stress that the cycle $\bar{\mathscr{D}}$ is not the strict basin of attraction of any stable equilibrium point, but, rather, it has a several-well structure: it contains in its interior, beyond C, the equilibria $R_{2,\perp}$ and $R_{2,\parallel}$; moreover, it contains the internal saddles S_4. The difference w.r.t the previous case of $l<\tilde{L}$ [where we had to consider the cycle $\bar{B}(C)$ in place of $\bar{\mathscr{D}}$] is that now not all the downhill paths emerging from $\sigma\in\bar{\mathscr{D}}$ end in C and the system, before escaping from $\bar{\mathscr{D}}$, will typically make many transitions back and forth between $\bar{B}(C)$ and $\bar{B}(R_{2,\perp})$, $\bar{B}(R_{2,\parallel})$ through S_4.

We consider now the frame C and suppose that $\tilde{L}\leqslant l<l^*$ and $m^*(l)\leqslant m<\tilde{M}-2$. We have $H(S_1)>H(S_{5,\parallel})$, $H(S_4)<H(S_{5,\parallel})$. With the usual arguments one can prove that

$$\min_{\sigma\in\partial\mathscr{D}} H(\sigma)=H(S_{5,\parallel}),\qquad U(\mathscr{D})=S_{5,\parallel}\tag{4.17}$$

Hence the frame C is supercritical and the system starting from C will hit $+\underline{1}$ in a typical time

$$\exp(\beta\{[2J-(h-\lambda)]-(h-\lambda)(m+1)+[2J-(h+\lambda)]\})$$

In the case $\tilde{L}<l<l^*$ and $m^*(l)<\tilde{M}-2\leqslant m$ it can be proved that $\min_{\sigma\in\partial\mathscr{D}} H(\sigma)=H(S_{5,\parallel})$, $U(\mathscr{D})=S_{5,\parallel}$, and $H(S_{5,\parallel})<H(S_4)$. Hence the frame is supercritical, but the typical escape time is $e^{\beta[2J-(h-\lambda)]}$. We remark that in this case $H(S_4)<H(S_1)<H(S_{5,\perp})$ and $H(S_{5,\parallel})<H(S_4)$ (see Fig. 26). Hence the set

$$\bar{\mathscr{D}}:=\{\sigma\in\mathscr{D};\ H(\sigma)\leqslant H(S_4)\}$$

is a *generalized* cycle like the set Q_1 defined in ref. 15. The set $\bar{\mathscr{D}}$ is a set of cycles communicating through the minimal saddles in S_4. Starting from $\mathscr{D}$, the system, before leaving $\mathscr{D}$ will not necessarily visit all the cycles contained in $\bar{\mathscr{D}}$ with energy less than $H(S_4)$.

The system can leave B either through $S_{4,\perp}$ or through $S_{4,\parallel}$. In the first case it will enter into $B_{2,\perp}$ visiting all the configurations of the cycle:

$$\bar{B}_{2,\perp}:=\{\sigma\in B_{2,\perp}:\ H(\sigma)<H(S_4)\}$$

before leaving $B_{2,\perp}$ and passing again through $S_{4,\perp}$. In the second case it will directly get out of $\mathscr{D}$.

In the case $l^*\leqslant l<\tilde{M}-2$ and $m<\tilde{M}-2$ we have that $\min_{\sigma\in\partial\mathscr{D}} H(\sigma)=H(S_{5,\parallel})$, $U(\mathscr{D})=S_{5,\parallel}$, and $H(S_{5,\parallel})>H(S_4)$. Hence the frame is supercritical and the typical escape time is

$$\exp(\beta\{[2J-(h-\lambda)]-(h-\lambda)(m+1)+[2J-(h+\lambda)]\})$$

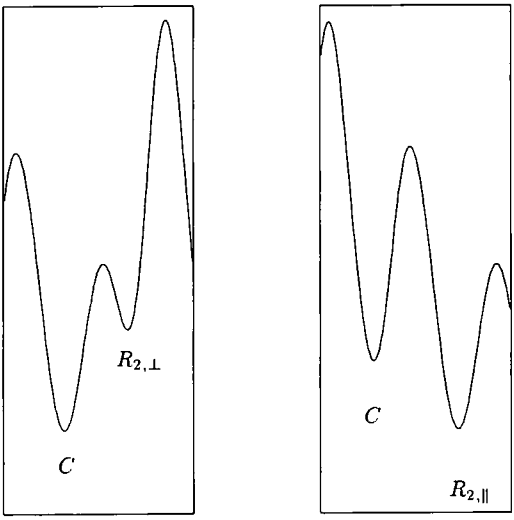


Fig. 26.

In this case the most important inequalities are

$$(h+\lambda)(l-1) > [2J-(h-\lambda)] - (h-\lambda)(l+1) + [2J-(h+\lambda)]$$
$$> [2J-(h-\lambda)] - (h-\lambda)(m+1) + [2J-(h+\lambda)] \qquad (4.18)$$

we remark that

$$H(S_1) - H(C) = (h+\lambda)(l-1)$$
$$H(S_{5,\perp}) - H(C) = [2J-(h-\lambda)] - (h-\lambda)(l+1) + [2J-(h+\lambda)]$$
$$H(S_{5,\parallel}) - H(C) = [2J-(h-\lambda)] - (h-\lambda)(m+1) + [2J-(h+\lambda)]$$

In the case $l^* \leqslant l < \tilde{M}-2$ and $\tilde{M}-2 \leqslant m$ we have that $\min_{\sigma \in \partial \mathscr{D}} H(\sigma) = H(S_{5,\parallel})$ and $H(S_{5,\parallel}) < H(S_4)$. Hence the frame is supercritical and the typical escape time is $e^{\beta[2J-(h-\lambda)]}$. In this case we have that $\mathscr{D}$ contains again a generalized cycle.

We remark that in the supercritical cases discussed above, namely for $l < \tilde{M}-2$, the growth of a rectangular frame is asymmetric. The frame grows in a direction parallel to its shortest side toward a squared frame. Notice that the same tendency to be attracted by a squared shape is present also in the contraction of a subcritical frame, which, as we have seen above, prefers to shrink in the direction orthogonal to its smallest side.

Finally we consider the case $l > \tilde{M} - 2$. It can be proved that $H(S_{5,\parallel}) < H(S_{5,\perp}) < H(S_4) < H(S_1)$; hence the frame is supercritical and the typical time is $e^{\beta[2J-(h-\lambda)]}$. In this case the growth process is symmetric, similar to what happens in the stochastic Ising model for *any* supercritical rectangle. This concludes the proof of Proposition 4.1. ▮

Remark. In the following, to avoid lengthy repetitions, we will often use short expressions like *the external rectangle shrinks in a direction perpendicular to its shortest sides* instead of "by a comparative analysis of the possible barriers of energy, namely looking at the set of minimal saddles of a suitable (possibly generalized) basin of attraction, we know that with a probability tending to one as β tends to infinity *the external rectangle shrinks in a direction perpendicular to its shortest sides.*"

In Proposition 4.1 we have stated conditions of subcriticality and supercriticality for frames; now we state similar results for birectangles.

Proposition 4.2. Let us consider a birectangle $R := R(L_1, L_2; M_1, M_2)$, let $L := \min\{L_1, L_2\}$, $\hat{L} := \max\{L_1, L_2\}$, $M := \min\{M_1, M_2\}$ and $\hat{M} := \max\{M_1, M_2\}$. If one of the conditions

1. $L < L^*, M < M^*$
2. $L \geq L^*, M < \tilde{M}, \hat{L} + 2 < \tilde{M}, L < \tilde{L}$
3. $L \geq L^*, M < \tilde{M}, \hat{L} + 2 < \tilde{M}, \tilde{L} \leq L < l^*, \hat{L} < m^*(L)$
4. $L \geq L^*, M < \tilde{M}, \hat{L} + 2 \geq \tilde{M}, M - 2 < \tilde{L}$
5. $L \geq L^*, M < \tilde{M}, \hat{L} + 2 \geq \tilde{M}, \tilde{L} \leq M - 2 < l^*, \hat{L} < m^*(M-2)$

is satisfied, then

$$\lim_{\beta \to \infty} P_R(\tau_{-\underline{1}} < \tau_{+\underline{1}}) = 1$$

If one of the conditions

6. $L \geq L^*, \tilde{M} \leq M < M^*$
7. $M \geq M^*$
8. $L \geq L^*, M < \tilde{M}, \hat{L} + 2 < \tilde{M}, L \geq l^*$
9. $L \geq L^*, M < \tilde{M}, \hat{L} + 2 < \tilde{M}, \tilde{L} \leq L < l^*, \hat{L} \geq m^*(L)$
10. $L \geq L^*, M < \tilde{M}, \hat{L} + 2 \geq \tilde{M}, M - 2 \geq l^*$
11. $L \geq L^*, M < \tilde{M}, \hat{L} + 2 \geq \tilde{M}, \tilde{L} \leq M - 2 < l^*, \hat{L} \geq m^*(M-2)$

is satisfied, then

$$\lim_{\beta \to \infty} P_R(\tau_{+\underline{1}} < \tau_{-\underline{1}}) = 1$$

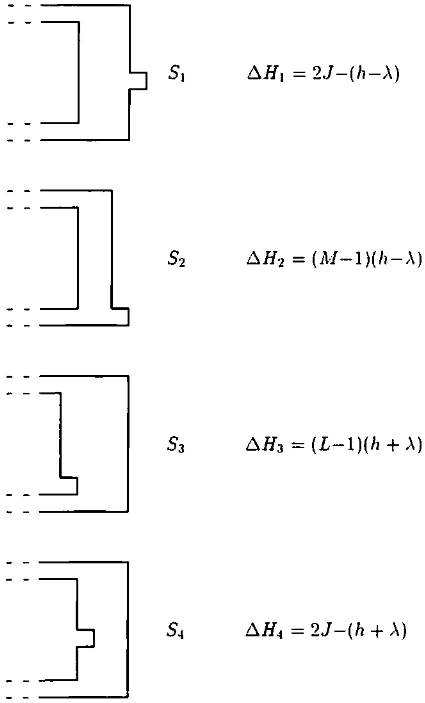


Fig. 27.

Proof. Without loss of generality we can assume $M=M_1$. Let us denote by $B:=B(R)$ the basin of attraction of R and by ∂B its boundary; first of all we have to find the minimum of the energy on the boundary ∂B. We examine all the uphill paths starting from R, but the relevant ones are those made of steps of the kinds (2, 2), (3, 2), (5, 1), and (10, 1) (see Figs. 16 and 21). The boundary configurations S_1, S_2, S_3 and S_4 reached by the uphill paths described above are represented in Fig. 27, together with the energy differences $\Delta H_i = H(S_i) - H(R)$ $\forall i = 1,..., 4$. We remark that certainly if both the shortest sides of the external rectangle are not "free," then at least one of the longest sides will be free; in this case $\Delta H_2 = (\hat{M}-1)(h-\lambda)$.

Now we consider case 1: as a consequence of the subcriticality of the internal and the external rectangle one has

$$
\begin{aligned}
L<L^* &\Rightarrow (L-1)(h+\lambda)<2J-(h+\lambda)<2J-(h-\lambda) \\
M<M^* &\Rightarrow (M-1)(h-\lambda)<2J-(h-\lambda)
\end{aligned}
\tag{4.19}
$$

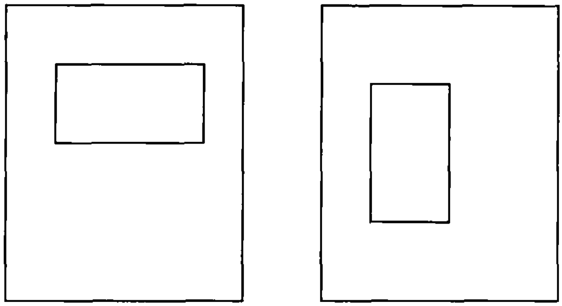


Fig. 28.

But we cannot say anything about the inequality $(L-1)(h+\lambda) \gtrless (M-1)(h-\lambda)$, without specifying more conditions on L and M; therefore the minimum of the energy on ∂B is given either by S_2 or S_3, depending on the values of L and M. Thus, the birectangle R is always subcritical and we can express the typical time needed by the system starting from R to hit $-\underline{1}$ as

$$\max\{e^{\beta(L-1)(h+\lambda)}, e^{\beta(M-1)(h-\lambda)}\}$$

Similar results are obtained if one supposes that the shortest side of the external rectangle is not "free."

Now, we suppose $L \geqslant L^*, M < \tilde{M}$, and $\hat{L}+2 \leqslant M$ (see Fig. 28): the internal rectangle is supercritical, namely $2J-(h+\lambda) < (L-1)(h+\lambda)$, and the external one is subcritical, namely $(M-1)(h-\lambda) < 2J-(h-\lambda)$; moreover,

$$M < \tilde{M} \Rightarrow (M-1)(h-\lambda) < 2J-(h+\lambda) \tag{4.20}$$

Then the minimum of the energy on the boundary ∂B is S_2. The external rectangle shrinks in a direction perpendicular to its shortest sides until it becomes a squared rectangle, then the shrinking process goes on in both directions until the frame $C(L_1, L_2)$ is reached in a typical time $e^{\beta(M-1)(h-\lambda)}$.

Even in the case $L \geqslant L^*, M < \tilde{M}$, and $M < \hat{L}+2 < \tilde{M}$ (the longest sides of the internal and the external rectangle are necessarily parallel) the system, starting from R, reaches the frame $C(L_1, L_2)$. Indeed the external rectangle shrinks along the direction perpendicular to its shortest sides until this process is stopped by the internal rectangle (see Fig. 29). In other words, this appears when the configuration $R(L, \hat{L}; M, \hat{L}+2)$ is reached (we have supposed, without loss of generality, that $L_1 = L$). At this point

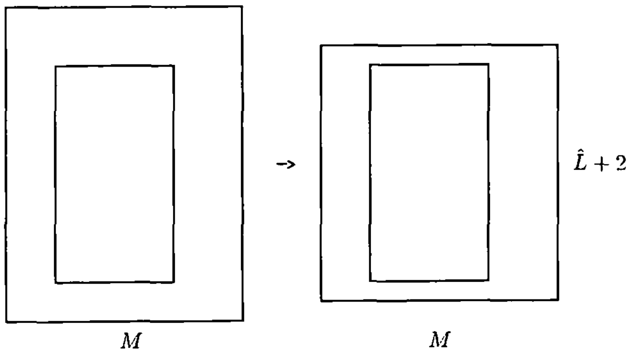


Fig. 29.

the external rectangle will begin to shrink along the direction perpendicular to its longest sides, because $\hat{L}+2<\tilde{M}$ and then $(\hat{L}+1)(h-\lambda)<2J-(h+\lambda)$. Hence, the system, starting from R, reaches the frame $C(L_1, L_2)$ in a typical time

$$\max\{e^{\beta(\hat{L}+1)(h-\lambda)}, e^{\beta(M-1)(h-\lambda)}\}=e^{\beta(\hat{L}+1)(h-\lambda)}$$

Then, we can conclude that in cases 2 and 3 the frame $C(L_1, L_2)$ is eventually reached, but $C(L_1, L_2)$ is subcritical, hence R is subcritical as well. For similar reasons in cases 8 and 9 the birectangle R is supercritical.

The typical shrinking time is given by

$$\max\{e^{\beta(M-1)(h-\lambda)}, e^{\beta(L-1)(h+\lambda)}\} \qquad \text{if} \quad \hat{L}+2\leqslant M$$
$$\max\{e^{\beta(\hat{L}+1)(h-\lambda)}, e^{\beta(L-1)(h+\lambda)}\} \qquad \text{if} \quad \hat{L}+2> M$$

With similar arguments it can be shown that in cases 4 and 5 the system, starting from R, hits $C(M-2, \hat{L})$ in a typical time $e^{2J-(h+\lambda)}$. Hence, the birectangle R is subcritical and the typical shrinking time is $e^{2J-(h+\lambda)}$. In the cases 10 and 11 the birectangle R is supercritical, as a consequence of the supercriticality of the frame $C(M-2, \hat{L})$.

With arguments similar to those used before it can also be seen that in case 6 the birectangle is supercritical, since it first evolves toward the frame $C(M-2, \hat{M}-2)$, which is a supercritical frame since $\hat{M}-2>l^*$ with our choice of the parameters.

Finally, in case 7, the birectangle is easily seen to be supercritical. Indeed it follows from an argument similar to the corresponding one valid for the standard Ising model that starting from a configuration with $M\geqslant M^*$, we get $\underline{0}$ before $+\underline{1}$ in a time of order $e^{\beta[2J-(h-\lambda)]}$ with high probability for large β. Then, starting from $\underline{0}$, we typically follow an Ising-like nucleation path[13, 19] leading to $+\underline{1}$ through the saddles $\mathscr{S}(\underline{0}, +\underline{1})$.

These saddles are given by configurations with precisely one cluster of pluses (in the sea of zeros); this cluster is given by a rectangle $L^* \times (L^* - 1)$ with a unit-square protuberance attached to one of its longest sides. It is immediate to verify that

$$H(\mathscr{S}(\underline{0}, +\underline{1})) < H(\mathscr{P})$$

See the definition of $\mathscr{P}$ in Section 2. The proof of Proposition 4.2 is complete. ∎

We consider now a plurirectangle R. We denote by M_1 and M_2 the lengths of the sides of the external rectangle, by $L_{1,i}$ and $L_{2,i} \forall i = 1,..., k^+$ the lengths of the sides of the k^+ internal rectangles R_i^+, and we define $M := \min\{M_1, M_2\}$ and $L_i := \min\{L_{1,i}, L_{2,i}\} \forall i = 1,..., k^+$. In order to state conditions of subcriticality and supercriticality for such configurations, we must introduce the rectangle R^+ defined as the rectangular envelope of the union of all the internal supercritical rectangles. We denote by L_{1,R^+} and L_{2,R^+} the lengths of its sides and we define $L_{R^+} := \min\{L_{1,R^+}, L_{2,R^+}\}$ and $\hat{L}_{R^+} := \max\{L_{1,R^+}, L_{2,R^+}\}$. Suppose that $\exists i \in \{1, 2,..., k^+\}$ such that $L_i \geqslant L^*$; we denote by $\bar{R}$ the birectangle obtained by removing all the internal rectangles and by filling up with plus spin the rectangle R^+. Finally we state the following proposition.

Proposition 4.3. If one of the two conditions

1. $L_i < l^* \; \forall i = 1,..., k^+$ and $M < M^*$
2. $\exists i \in \{1, 2,..., k^+\}$ such that $L_i \geqslant L^*$ and $\bar{R}$ is subcritical

is satisfied, then

$$\lim_{\beta \to \infty} P_R(\tau_{-\underline{1}} < \tau_{+\underline{1}}) = 1$$

Proof. Let us consider case 1: we prove Proposition 4.3 by describing the shrinking process.

First of all, the internal rectangles whose sides are such that $(L_i - 1)(h + \lambda) < (M - 1)(h - \lambda)$ shrink in a typical time $\exp[\beta(L_i - 1)(h + \lambda)]$. We denote by $R^{(1)}$ the rectangular envelope of the union of all the "surviving" rectangles $R_i^{(1)} \; \forall i \in I^{(1)} \subset \{1,..., k^+\}$ and by $\hat{L}^{(1)}$ its longest side.

At this point the external rectangle starts shrinking (if it can). If $\hat{L}^{(1)} \leqslant M - 2$, this contraction ends when the external rectangle reaches $R^{(1)}$ (see Fig. 30).

Let us define $L_{\min} := \min_{i \in I^{(1)}}\{L_i\}$: the internal rectangle R_i^+ such that $L_i = L_{\min}$ starts shrinking and loses a slice of length $L_i = L_{\min}$. There are

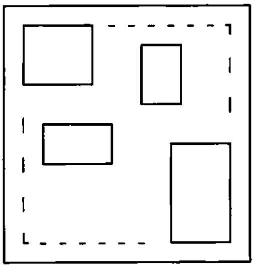


Fig. 30.

two possible situations (see Fig. 31): after this contraction the external rectangle has a "free" side or not. In the first case the external rectangle loses another slice and a configuration of the type described in Fig. 30 is reached. In the second case the internal rectangle goes on shrinking until it disappears, and a configuration like the one in Fig. 30 is reached, as well. In both cases the plurirectangle goes on shrinking by the mechanism described before until it disappears; hence in the case $\hat{L}^{(1)} \leqslant M-2$ the plurirectangle R is subcritical.

Now, we consider the case $\hat{L}^{(1)} > M-2$. During the second phase of the contraction the system reaches a configuration characterized by an external rectangle whose sides are M and $\hat{L}^{(1)}+2$. The "free" side of the external rectangle is eventually $\hat{L}^{(1)}+2$. If $(\hat{L}^{(1)}+1)(h-\lambda) < (L_i-1)(h+\lambda)$ $\forall i \in I^{(1)}$, the external rectangle shrinks in a direction perpendicular to its "free" side until it reaches $R^{(1)}$; and then the shrinking goes on as we have described before. If there exists an internal rectangle R_i^+ such that $(\hat{L}^{(1)}+1)(h-\lambda) > (L_i-1)(h+\lambda)$, it disappears before anything else can happen. Then the contraction goes on as described before. In conclusion we have proved that in case 1 the plurirectangle R is subcritical.

In case 2 the proof of Proposition 4.3 can be achieved with arguments similar to those used in case 1. ▮

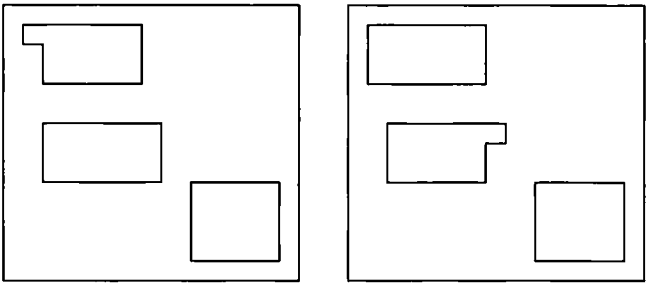


Fig. 31.

5. COMPARISON BETWEEN SPECIAL SADDLES

Let us consider a subcritical frame or birectangle; we say that such a configuration is *almost supercritical* iff it can be transformed into a supercritical minimum by attaching to one of its internal or external sides a whole slice. By attaching a slice to an internal or external side of a birectangle (or, in particular, of a frame) we mean transforming from -1 to 0 the value of the spins in the row or column adjacent externally to this side. "Removing a slice" is the inverse operation of "attaching a slice."

Let us consider a supercritical frame or birectangle; we say that such a configuration is *just supercritical* iff it can be transformed into a subcritical minimum by removing a whole slice from one of its internal or external sides.

Let us consider an almost-supercritical frame or birectangle; we denote by u the internal or external side such that by attaching to it a whole slice we obtain a supercritical configuration. We call *special saddle* the configuration obtained by attaching to u a plus unit protuberance if u is an internal side, or a zero unit protuberance if u is an external one.

Let us consider the set $\hat{\mathscr{P}} := (\mathscr{P}_1 \cup \mathscr{P}_2) \subset \Omega_\Lambda$ with $\mathscr{P}_1$ and $\mathscr{P}_2$ the set of special saddles shown in Fig. 32, where we have used the definition

$$\delta := l^* - \frac{2J-(h-\lambda)}{h} \in \,]0, 1[\tag{5.1}$$

We state the following lemma:

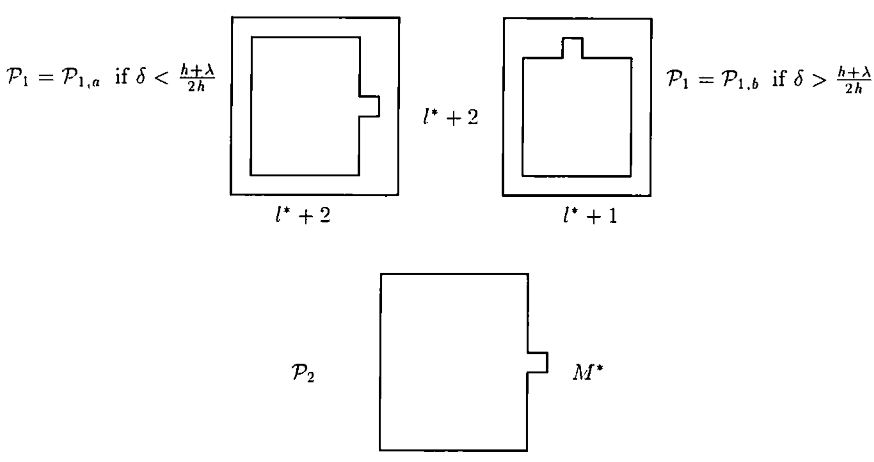


Fig. 32.

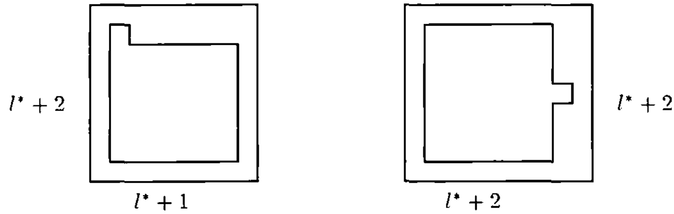


Fig. 33.

Lemma 5.1. For any special saddle $S \notin \hat{\mathscr{P}}$ there exists $S^* \in \hat{\mathscr{P}}$ such that

$$H(S) > H(S^*)$$

Before starting the proof, we observe that the frame $C(l^*, l^*)$ is supercritical and $C(l^* - 1, l^* - 1)$ is subcritical for any choice of the parameters λ and h; indeed it can be proved that

$$m^*(l^* - 1) \geqslant l^* \qquad \text{for any value of } h \text{ and } \lambda \tag{5.2}$$

[see (5.5)]. On the other hand, we remark that the criticality of the frame $C(l^* - 1, l^*)$ depends on the value of the real number δ defined in (5.1). By comparing the energies of the two configurations shown in Fig. 33 one can easily convince oneself that

$$C(l^* - 1, l^*) \qquad \text{subcritical} \quad \text{iff} \quad \delta < \frac{h + \lambda}{2h}$$

$$C(l^* - 1, l^*) \qquad \text{supercritical} \quad \text{iff} \quad \delta > \frac{h + \lambda}{2h}$$

we observe that $(h + \lambda)/2h \in]0, 1[$ if $h/\lambda > 1$. This explains the reason for the twofold definition of the configuration $\mathscr{P}_1$.

Proof of Lemma 5.1. Let us suppose that $\delta < (h + \lambda)/2h$. One can prove that for any l such that $\tilde{L} \leqslant l \leqslant l^* - 1$

$$m^*(l) \geqslant l^* + 1 \tag{5.3}$$

First of all we observe that $m^*(l)$ is a decreasing function of l, more precisely, one can easily prove that

$$m^*(l - 1) \geqslant m^*(l) + 1 \qquad \forall l \in [\tilde{L}, l^* - 1] \tag{5.4}$$

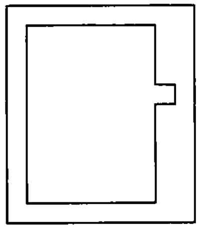


Fig. 34.

Therefore in order to get a lower bound to $m^*(l)$ it is sufficient to evaluate $m^*(l^*-1)$; with some algebra one can easily obtain

$$m^*(l^*-1) = l^* + \left[(1-\delta)\frac{2h}{h-\lambda} \right] \tag{5.5}$$

Then,

$$\delta < \frac{h+\lambda}{2h} \Rightarrow (1-\delta)\frac{2h}{h-\lambda} > 1 \Rightarrow m^*(l^*-1) \geqslant l^*+1$$

this completes the proof of inequality (5.3). We remark that the validity of (5.4) and (5.5) does not depend on the value of the real number δ.

Now, in order to prove Lemma 5.1, we have to examine all the possible special saddles.

Case C1. We consider the special saddle $C_1(m)$ in Fig. 34. It can be easily shown that $H(C_1(m-1))$ is an increasing function of $m \in [l^*, m^*(l^*-1)-1]$; indeed

$$H(C_1(m+1)) - H(C_1(m)) = (h+\lambda) - 2h\,\delta > 0$$

by virtue of the hypothesis $\delta < (h+\lambda)/2h$. Hence,

$$H(C_1(m)) \geqslant \mathscr{P}_1 \qquad \forall m \in [l^*, m^*(l^*-1)-1] \tag{5.6}$$

We observe that the equality is verified in (5.6) iff $m = l^*$, that is, $C_1(m) \equiv \mathscr{P}_1$.

Case C2. We consider the special saddles $C_{2,a}(l)$ and $C_{2,b}(l)$ in Fig. 35. We remark that the configuration obtained from $C_{2,b}(l)$ by removing the protuberance is subcritical because $m^*(l-1) \geqslant m^*(l)+1$.

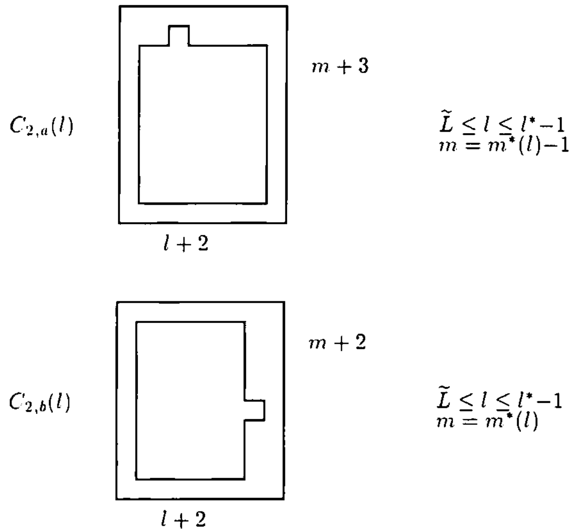


Fig. 35.

We have that $H(C_{2,a}(l)) < H(C_{2,b}(l))$; indeed, $H(C_{2,a}(l)) - H(C_{2,b}(l)) = (h+\lambda)(l - m^*(l))$ and $l - m^*(l) < 0$; the last inequality is a consequence of the fact that $l < l^*$ and of Eq. (5.2): $m^*(l) \geqslant m^*(l^* - 1) \geqslant l^* > l$.

We observe that $H(C_{2,a}(l))$ is a decreasing function of l:

$$H(C_{2,a}(l+1)) < H(C_{2,a}(l)) \qquad \forall l \in [\tilde{L}, l^* - 2] \tag{5.7}$$

Indeed, it is not difficult to show that

$$H(C_{2,a}(l+1)) - H(C_{2,a}(l)) = (h+\lambda) + 2h(l^* - l - \delta)(m^*(l+1) - m^*(l)) - 2h(m^*(l+1) - l^* + \delta)$$

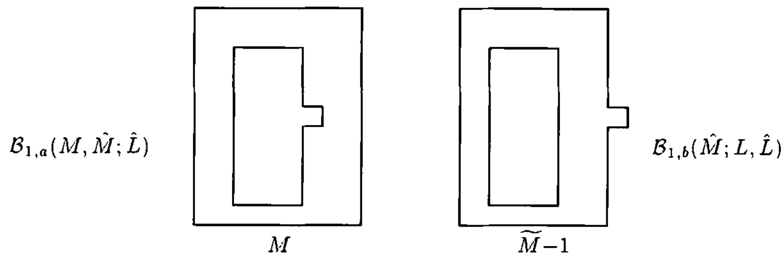
and by observing that $l^* - l - \delta > +1, m^*(l+1) - m^*(l) < -1$, and $m^*(l+1) - l^* + \delta < 0$ we obtain

$$H(C_{2,a}(l+1)) - H(C_{2,a}(l)) < (h+\lambda) - 2h = \lambda - h < 0$$

This completes the proof of the inequality (5.7).

Since $H(C_{2,a}(l))$ is a decreasing function, we have to compare the energy of the two configurations $C_{2,a}(l^* - 1)$ and $\mathscr{P}_1$; by a direct calculation one obtains $H(C_{2,a}(l^* - 1)) > H(\mathscr{P}_1)$.

Case B1. We consider the special saddles $\mathscr{B}_{1,a}(M, \hat{M}; \hat{L})$ and $\mathscr{B}_{1,b}(\hat{M}; L, \hat{L})$ in Fig. 36 (here and in the following we use the notation



(a) (b)

Fig. 36. (a) The internal horizontal dimension is L^*-1 and the vertical one $\hat{L}$ is such that $\hat{L} \geqslant L^*$. The external vertical dimension is $\hat{M}$ and the horizontal one M is such that $\tilde{M} \leqslant M < M^*$. If we choose the parameters h and λ such that $\tilde{M} = M^*$, then the special saddle (a) does not exist. (b) The external horizontal dimension is $\tilde{M}-1$ and the vertical one is $\hat{M}$. The internal dimensions L and $\hat{L}$ are such that $L \geqslant L^*$ and by removing the external unit protuberance one obtains a subcritical birectangle.

introduced in Proposition 4.2 to label the internal and external sides; we use L, $\hat{L}$, M, and $\hat{M}$ to denote the dimensions of the birectangle obtained by removing the unit protuberance of the special saddle).

We observe that

$$H(\mathscr{B}_{1,a}(M, \hat{M}; \hat{L})) \geqslant H(\mathscr{B}_{1,a}(\tilde{M}, \tilde{M}; L^*)) \tag{5.8}$$

for every possible choice of the positive integer numbers $\hat{M}$, M and $\hat{L}$. This is an obvious consequence of the fact that $L = L^*-1 < L^*$ and $M < M^*$.

Now, we transform $\mathscr{B}_{1,a}(\tilde{M}, \tilde{M}; L^*)$ into $\mathscr{P}_1$ in several steps and we evaluate the energy cost ΔH_i of each step.

- $\mathscr{B}_{1,a}(\tilde{M}, \tilde{M}; L^*) \to R(L^*-1, L^*; \tilde{M}, \tilde{M})$, $\Delta H_1 := H(R(L^*-1, L^*; \tilde{M}, \tilde{M})) - H(\mathscr{B}_{1,a}(\tilde{M}, \tilde{M}; L^*)) = -[2J-(h+\lambda)]$.

- $R(L^*-1, L^*; \tilde{M}, \tilde{M}) \to R(L^*-1, L^*; l^*+2, l^*+2)$, $\Delta H_2 < 0$ because the external rectangle is subcritical and $\tilde{M} > l^*+2$ [see inequalities (3.17)].

- $R(L^*-1, L^*; l^*+2, l^*+2) \to R(L^*, L^*; l^*+2, l^*+2)$, $\Delta H_3 < 0$ because a whole internal slice of length L^* has been attached to the internal (relatively) supercritical rectangle.

- $R(L^*, L^*; l^*+2, l^*+2) \to R(l^*-1, l^*; l^*+2, l^*+2)$, $\Delta H_4 < 0$ because the internal rectangle is supercritical and $L^* < l^*-1$.

- $R(l^*-1, l^*; l^*+2, l^*+2) \to \mathscr{P}_1$, $\Delta H_5 = 2J-(h+\lambda)$.

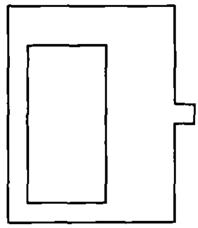


Fig. 37. The internal dimensions L and $\hat{L}$ are such that the birectangle obtained by removing the zero unit protuberance is subcritical. The external dimension $M^* - 1$ and M are such that $M \geqslant M^*$.

One has that $\sum_{i=1}^{5} \Delta H_i < 0$, hence $H(\mathscr{B}_{1,a}(\tilde{M}, \tilde{M}; L^*)) > H(\mathscr{P}_1)$. This inequality and (5.8) lead to the conclusion that

$$H(\mathscr{B}_{1,a}(M, \hat{M}; \hat{L})) > \mathscr{P}_1 \tag{5.9}$$

for every possible choice of $\hat{M}$, M and $\hat{L}$.

In order to characterize the special saddle $\mathscr{B}_{1,b}(\hat{M}; L, \hat{L})$, we have to distinguish two possible cases.

Case (i). $\hat{L} + 2 \geqslant \tilde{M}$: The birectangle $R(L, \hat{L}; \tilde{M} - 1, \hat{M})$, obtained from $\mathscr{B}_{1,b}(\hat{M}; L, \hat{L})$ by removing the external unit protuberance, must be subcritical. Then, by virtue of Proposition 4.2, one can say that must necessarily have $(\tilde{M} - 1) - 2 \leqslant l^* - 1$, that is $\tilde{M} \leqslant l^* + 2$. This is absurd [see inequalities (3.17)]. Then we can conclude that there does not exist a special saddle $\mathscr{B}_{1,b}(\hat{M}; L, \hat{L})$ such that $\hat{L} + 2 \geqslant \tilde{M}$.

Case (ii). $\hat{L} + 2 < \tilde{M}$: The internal rectangle $L \times \hat{L}$ must be contained in the rectangle $L \times (m^*(L) - 1)$, otherwise the birectangle $R(L, \hat{L}; \tilde{M} - 1, \hat{M})$ would be supercritical. Now we transform the special saddle $\mathscr{B}_{1,b}(\hat{M}; L, \hat{L})$ into $C_{2,a}(L+1)$ (notice that $L \geqslant L^* \Rightarrow L + 1 \geqslant \hat{L}$) and we show that the energy is reduced.

- $\mathscr{B}_{1,b}(\hat{M}; L, \hat{L}) \to R(L, \hat{L}; \tilde{M} - 1, \hat{M})$, $\Delta H_1 = -[2J - (h - \lambda)]$.
- $R(L, \hat{L}; \tilde{M} - 1, \hat{M}) \to R(L+1, \hat{L}; \tilde{M} - 1, \hat{M})$, $\Delta H_2 \leqslant 0$ because $\hat{L} \geqslant L \geqslant L^*$.
- $R(L+1; \hat{L}; \tilde{M} - 1, \hat{M}) \to R(L+1, m^*(L+1) - 1; \tilde{M} - 1, \hat{M})$, $\Delta H_3 \leqslant 0$ because $L \geqslant L^*$ and $\hat{L} \leqslant m^*(L+1) - 1$.
- $R(L+1, m^*(L+1) - 1; \tilde{M} - 1, \hat{M}) \to R(L+1, m^*(L+1) - 1; L+2, m^*(L+1) + 2)$, $\Delta H_4 < 0$ because the external rectangle is subcritical and $L + 2 < \tilde{M} - 1$.
- $R(L+1, m^*(L+1) - 1; L+2, m^*(L+1) + 2) \to C_{2,a}(L+1)$, $\Delta H_5 = 2J - (h + \lambda)$.

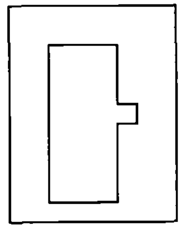


Fig. 38. The internal horizontal dimension is $l^* - 1$ and the vertical one $\hat{L}$ is such that $l^* \leqslant \hat{L} < \tilde{M} - 2$ and $\hat{L} < m^*(l^* - 1)$. The external vertical dimension is $\hat{M}$ and the horizontal one is $M < \tilde{M}$.

Hence,

$$H(\mathscr{B}_{1,b}(\hat{M}; L, \hat{L})) > H(C_{2,a}(L+1)) > H(\mathscr{P}_1) \tag{5.10}$$

for every possible choice of the dimensions $\hat{M}$, L, and $\hat{L}$.

Case B2. We consider the special saddle $\mathscr{B}_2(\hat{M}; L, \hat{L})$ in Fig. 37. Two possible cases must be considered.

Case (i). $M^* > \tilde{M}$: The internal rectangle is subcritical, hence by removing it we obtain a configuration at lower energy. Then, by means of arguments similar to those used in the case of the standard Ising model (see e.g., ref. 13), one can prove that

$$H(\mathscr{B}_2(\hat{M}; L, \hat{L})) \geqslant H(\mathscr{P}_2) \tag{5.11}$$

where the equality stands iff $\mathscr{B}_2(\hat{M}; L, \hat{L}) \equiv \mathscr{P}_2$.

Case (ii). $M^* = \tilde{M}$: See the discussion for the special saddle $\mathscr{B}_{1,b}(\hat{M}; L, \hat{L})$.

Case B3. We consider the special saddle $\mathscr{B}_3(M, \hat{M}; \hat{L})$ in Fig. 38. One can easily prove that $H(\mathscr{B}_3(M, \hat{M}; \hat{L})) \geqslant H(C_1(\hat{L}))$ by virtue of the inequalities $M < M^*$ and $\hat{L} + 2 < \tilde{M} \leqslant M^*$.

Hence, we conclude that

$$H(\mathscr{B}_3(M, \hat{M}; \hat{L})) \geqslant H(C_1(\hat{L})) \geqslant H(\mathscr{P}_1) \tag{5.12}$$

for every possible choice of the dimensions M, $\hat{M}$ and $\hat{L}$. We observe that in (5.12) the equality holds iff $\mathscr{B}_3(M, \hat{M}; \hat{L}) \equiv \mathscr{P}_1$.

Case B4. We consider the special saddles $\mathscr{B}_{4,a}(M, \hat{M}; L)$ and $\mathscr{B}_{4,b}(M, \hat{M}; L)$ in Fig. 39. First of all we observe that

$$H(\mathscr{B}_{4,a}(M, \hat{M}; L)) \geqslant C_{2,a}(L), \qquad H(\mathscr{B}_{4,b}(M, \hat{M}; L)) \geqslant C_{2,b}(L) \tag{5.13}$$

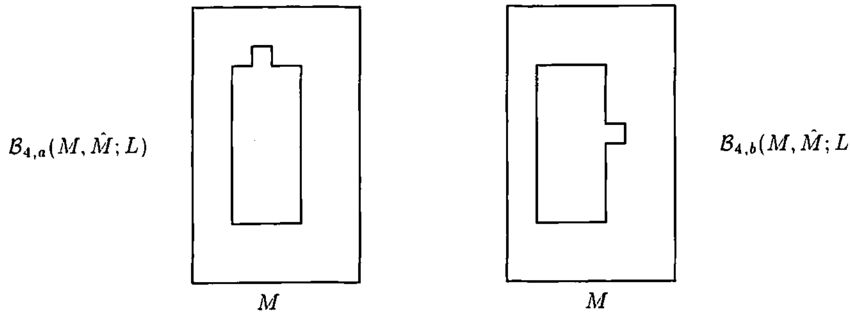


Fig. 39. (a) The internal horizontal dimension L is such that $\tilde{L} \leqslant L \leqslant l^* - 1$. The vertical one is $\hat{L} = m^*(L) - 1$ and it is such that $\hat{L} + 3 < \tilde{M}$. The external vertical dimension is $\hat{M}$ and the horizontal one M is such that $M < \tilde{M}$. (b) The external dimensions are like those in (a). The internal horizontal dimension $L - 1$ is such that $\tilde{L} \leqslant L - 1 \leqslant l^* - 2$. The vertical one is $\hat{L} = m^*(L)$ and it is such that $\hat{L} + 2 < \tilde{M}$. We remark that for certain choices of the parameters h and λ the configurations in (a) and (b) cannot be considered; it could be, indeed, $m^*(L) \geqslant \tilde{M} - 2$.

for every possible choice of M, $\hat{M}$, and L. Conditions (5.13) are a consequence of the fact that $M < \tilde{M} \leqslant M^*$ and $\hat{L} + 3 < \tilde{M} \leqslant M^*$ in both cases. The equalities are satisfied in (5.13) iff $\mathcal{B}_{4,a}(M, \hat{M}; L) \equiv C_{2,a}(L)$ or $\mathcal{B}_{4,b}(M, \hat{M}; L) \equiv C_{2,b}(L)$.

Now, by arguments similar to those used in the discussion of Case C2, we can prove that $H(\mathcal{B}_{4,a}(M, \hat{M}; L)) > H(\mathcal{P}_1)$ and $H(\mathcal{B}_{4,b}(M, \hat{M}; L)) > H(\mathcal{P}_1)$.

Case B5. We consider the special saddle $\mathcal{B}_5(\hat{M}; L, \hat{L})$ in Fig. 40.

Now we transform the special saddle $\mathcal{B}_5(\hat{M}; L, \hat{L})$ into $C_1(\hat{L})$ and we show that the energy is reduced.

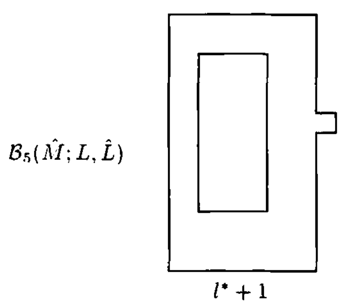


Fig. 40. The internal horizontal dimension L is such that $L^* \leqslant L \leqslant l^* - 1$ and the vertical one $\hat{L}$ is such that $\tilde{M} - 2 \leqslant \hat{L} < m^*(l^* - 1)$. The external vertical dimension is $\hat{M}$. We remark that this special saddle does not exist if we choose the parameters h and λ such that $m^*(l^* - 1) \leqslant \tilde{M} - 2$.

- $\mathscr{B}_5(\hat{M}; L, \hat{L}) \to R(L, \hat{L}; l^*+2, \hat{M})$, $\Delta H_1 = -(h-\lambda)(\hat{M}-1)$.
- $R(L, \hat{L}; l^*+2, \hat{M}) \to R(l^*-1, \hat{L}; l^*+2, \hat{M})$, $\Delta H_2 \leqslant 0$ because $L \geqslant \tilde{M}-2 > L^*$ and $L \leqslant l^*-1$.
- $R(l^*-1, \hat{L}; l^*+2, \hat{M}) \to R(l^*-1, \hat{L}; l^*+2, \hat{L}+2)$, $\Delta H_3 \leqslant 0$ since $l^*+2 < M^*$.
- $R(l^*-1, \hat{L}; l^*+2, \hat{L}+2) \to C_1(\hat{L})$, $\Delta H_4 = 2J - (h+\lambda)$.

By a direct calculation it can be proved that $\Delta H_1 + \Delta H_4 \leqslant 0$; then $H(\mathscr{B}_5(\hat{M}; L, \hat{L})) > H(C_1(\hat{L})) \geqslant H(\mathscr{P}_1)$.

Case B6. We consider the special saddles $\mathscr{B}_{6,a}(M, \hat{M}; L)$ and $\mathscr{B}_{6,b}(M, \hat{M}; L)$ in Fig. 41.

Now, we transform the special saddle $\mathscr{B}_{6,a}(M, \hat{M}; L)$ into $C_{2,a}(M-2)$ and show that the energy is reduced.

- $\mathscr{B}_{6,a}(M, \hat{M}; L) \to \mathscr{B}_{6,a}(M, \hat{L}+3; L)$, $\Delta H_1 \leqslant 0$ since $M \leqslant l^*+1 < M^*$.
- $\mathscr{B}_{6,a}(M, \hat{L}+3; L) \to C_{2,a}(M-2)$, $\Delta H_2 \leqslant 0$ because $\hat{L} \geqslant \tilde{M}-3 \geqslant l^* > L^*$.

Hence, we conclude that $H(\mathscr{B}_{6,a}(M, \hat{M}; L)) \geqslant H(C_{2,a}(M-2)) > H(\mathscr{P}_1)$.

The special saddle $\mathscr{B}_{6,b}(M, \hat{M}; L)$ can be transformed into the configuration $C_{2,b}(M-1)$ reducing the energy.

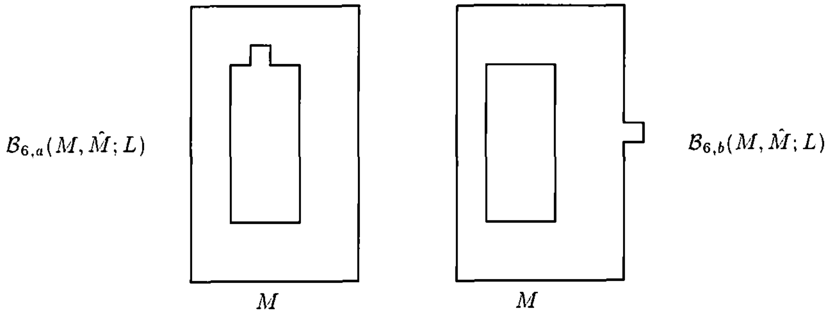


Fig. 41. (a) The internal horizontal dimension L is such that $L \geqslant L^*$. The vertical one is $\hat{L} = m^*(M-2)-1$ and it is such that $\hat{L}+3 \geqslant \tilde{M}$. The external vertical dimension is $\hat{M}$ and the horizontal one M is such that $\tilde{L} \leqslant M-2 \leqslant l^*-1$. (b) The internal horizontal dimension L is such that $L \geqslant L^*$. The vertical one is $\hat{L} = m^*(M-1)$ and it is such that $\hat{L}+2 \geqslant \tilde{M}$. The external vertical dimension is $\hat{M}$ and the external horizontal dimension M is such that $\tilde{L} \leqslant M-1 \leqslant l^*-1$. We remark that for certain choices of the parameters h and λ the configurations in (a) and (b) cannot be considered.

• $\mathscr{B}_{6,b}(M, \hat{M}; L) \to R(L, \hat{L}; M+1, \hat{M})$, $\Delta H_1 = -(h-\lambda)(\hat{M}-1)$.

• $R(L, \hat{L}; M+1, \hat{M}) \to R(L, \hat{L}; M+1, \hat{L}+2)$, $\Delta H_2 \leqslant 0$ since $M+1 \leqslant l^*+1 < M^*$.

• $R(L, \hat{L}; M+1, \hat{L}+2) \to R(M-2, \hat{L}; M+1, \hat{L}+2)$, $\Delta H_3 \leqslant 0$ because $\hat{L} \geqslant \tilde{M}-2 > L^*$.

• $R(M-2, \hat{L}; M+1, \hat{L}+2) \to C_{2,b}(M-1)$, $\Delta H_4 = 2J-(h+\lambda)$.

It is easily seen that $\Delta H_1 + \Delta H_4 < 0$, hence $H(\mathscr{B}_{6,a}(M, \hat{M}; L)) > H(C_{2,b}(M-1)) > H(\mathscr{P}_1)$.

This completes the proof of Lemma 5.1 in the case $\delta < (h+\lambda)/2h$. We suppose, now, $\delta > (h+\lambda)/2h$ and observe that in this case

$$m^*(l^*-1) = l^* \tag{5.14}$$

as follows from Eq. (5.5). In the sequel we will analyze all the cases that have to be discussed with arguments different from those used before.

Case C1. The special saddle $C_1(m)$ with $l^* \leqslant m \leqslant m^*(l^*-1)-1$ cannot be considered, since $m^*(l^*-1)-1 = l^*-1$ [see (5.14)].

Case C2. We proved above that $C_{2,a}(l^*-1)$ is the special saddle with lowest energy among $C_{2,a}(L)$ and $C_{2,b}(L)$. This result is not dependent on the value of the real number δ. Hence, one can say

$$H(C_{2,b}(l)) > H(C_{2,a}(l)) \geqslant H(C_{2,a}(l^*-1)) = H(\mathscr{P}_1)$$

we remark that in the case $\delta > (h+\lambda)/2h$ the special saddle $C_{2,a}(l^*-1)$ and the global saddle $\mathscr{P}_1$ coincide.

Case B1. In order to prove that $H(\mathscr{B}_{1,a}(M, \hat{M}; \hat{L})) > H(\mathscr{P}_1)$, we have to consider two different cases.

Case (i). $\hat{L} \geqslant l^*-1$: We transform the special saddle $\mathscr{B}_{1,a}(M, \hat{M}; \hat{L})$ into $\mathscr{P}_1$ and we prove that the energy is reduced.

• $\mathscr{B}_{1,a}(M, \hat{M}; \hat{L}) \to R(L^*-1, \hat{L}; M, \hat{M})$, $\Delta H_1 = -[2J-(h+\lambda)]$.

• $R(L^*-1, \hat{L}; M, \hat{M}) \to R(L^*-1, l^*-1; M, \hat{M})$, $\Delta H_2 \leqslant 0$ since $L^*-1 < L^*$ and $\hat{L} \geqslant l^*-1$.

• $R(L^*-1, l^*-1; M, \hat{M}) \to R(l^*-1, l^*-1; M, \hat{M})$, $\Delta H_3 < 0$ because $l^*-1 \geqslant L^*$.

• $R(l^*-1, l^*-1; M, \hat{M}) \to R(l^*-1, l^*-1; l^*+2, l^*+1)$, $\Delta H_4 < 0$ since $M < M^*$ and $M > l^*+2$.

• $R(l^*-1, l^*-1; l^*+2, l^*+1) \to \mathscr{P}_1$, $\Delta H_5 = 2J-(h+\lambda)$.

We conclude that $H(\mathscr{B}_{1,a}(M, \hat{M}; \hat{L})) > H(\mathscr{P}_1)$ since $\sum_{i=1}^{5} \Delta H_i < 0$.

Case (ii). $L^* \leqslant \hat{L} < l^* - 1$: First we notice that this case can be considered only if $l^* - 1 > L^*$. Now we transform $\mathscr{B}_{1,a}(M, \hat{M}; \hat{L})$ into $\mathscr{P}_1$.

- $\mathscr{B}_{1,a}(M, \hat{M}; \hat{L}) \to R(L^* - 1, \hat{L}; M, \hat{M})$, $\Delta H_1 = -[2J - (h + \lambda)]$.
- $R(L^* - 1, \hat{L}; M, \hat{M}) \to R(l^* - 1, \hat{L}; M, \hat{M})$, $\Delta H_2 < 0$ since $L^* - 1 < l^* - 1$ and $\hat{L} \geqslant L^*$.
- $R(l^* - 1, \hat{L}; M, \hat{M}) \to R(l^* - 1, l^* - 1; M, \hat{M})$, $\Delta H_3 < 0$ because $l^* - 1 \geqslant L^*$ and $\hat{L} < l^* - 1$.
- $R(l^* - 1, l^* - 1; M, \hat{M}) \to R(l^* - 1, l^* - 1; l^* + 2, l^* + 1)$, $\Delta H_4 < 0$ since $M < M^*$ and $M > l^* + 2$.
- $R(l^* - 1, l^* - 1; l^* + 2, l^* + 1) \to \mathscr{P}_1$, $\Delta H_5 = 2J - (h + \lambda)$.

Also in this case we conclude that $H(\mathscr{B}_{1,a}(M, \hat{M}; \hat{L})) > H(\mathscr{P}_1)$.

Finally, with arguments similar to those used in the case $\delta < (h + \lambda)/2h$ one can show that $H(\mathscr{B}_{1,b}(\hat{M}; L, \hat{L})) > H(C_{2,a}(L)) > H(\mathscr{P}_1)$.

Case B3. This case cannot be considered, because the inequalities $l^* \leqslant \hat{L} < m^*(l^* - 1)$ cannot be verified [see (5.14)].

Case B5. This case cannot be considered, because the inequalities $\tilde{M} - 2 \leqslant \hat{L} < m^*(l^* - 1)$ cannot be verified. Indeed, from (3.15) one has $l^* + 3 \leqslant \tilde{M}$; hence $l^* < \tilde{M} - 2$. Finally, $m^*(l^* - 1) = l^* \Rightarrow m^*(l^* - 1) < \tilde{M} - 2$.

The proof of Lemma 5.1 is now complete. ∎

6. THE SET $\mathscr{G}$ AND THE MINIMUM OF THE ENERGY ON $\partial\mathscr{G}$

In this section we define a set $\mathscr{G}$ of configurations which will play a basic role in the proof of our results. $\mathscr{G}$ will constitute an "upper estimate" of the generalized basin of attraction of $-\underline{1}$, in the sense that every subcritical configuration, that is, a configuration σ such that

$$\lim_{\beta \to \infty} P_\sigma(\tau_{-\underline{1}} < \tau_{+\underline{1}}) = 1 \tag{6.1}$$

will belong to $\mathscr{G}$; moreover, given any $\eta \in \mathscr{G}$, there exists a downhill path leading to a configuration σ satisfying (6.1). On the other hand, there are configurations $\eta \in \mathscr{G}$ which are supercritical in the sense that

$$\lim_{\beta \to \infty} P_\eta(\tau_{-\underline{1}} > \tau_{+\underline{1}}) = 1 \tag{6.2}$$

The crucial property of $\mathscr{G}$ will be that the minimum of the energy in its boundary $\partial\mathscr{G}$ will be given by $\mathscr{P}_1$ or $\mathscr{P}_2$.

We will see that this implies that for every configuration σ with sufficiently low energy $[H(\sigma) < \min\{H(\mathscr{P}_1), H(\mathscr{P}_2)\}]$, (6.1) is satisfied.

It is possible to find examples of configurations belonging to $\mathscr{G}$ which are potentially supercritical in the sense that (6.1) fails.

To construct $\mathscr{G}$, first of all we define a map $\mathscr{F}: \sigma \to \hat{\sigma} = \mathscr{F}\sigma$ with σ an acceptable configuration and $\hat{\sigma}$ a local minimum of the energy, such that the two following properties are satisfied:

$$\begin{aligned} &H(\hat{\sigma}) \leqslant H(\sigma) \\ &\sigma \prec \hat{\sigma} \end{aligned} \tag{6.3}$$

that is, the local minimum $\hat{\sigma}$ is bigger than σ and at a lower energy level. Then we define the set $\mathscr{G}$ as the set of configurations σ such that $\hat{\sigma}$ is subcritical, that is

$$P_{\hat{\sigma}}(\tau_{-\underline{1}} < \tau_{+\underline{1}}) \to 1 \qquad \text{as} \quad \beta \to \infty$$

Now we define the map $\mathscr{F}: \sigma \to \hat{\sigma}$; the definition is given in the following six steps. Let σ be an acceptable configuration:

(i) Starting from σ, we construct the configuration σ_1 by turning into zero all the minus spins of σ which have at least one plus spin among their nearest neighbor sites. We remark that $H(\sigma_1) \leqslant H(\sigma)$ (see Fig. 2) and $\sigma \prec \sigma_1$.

(ii) Let us denote by c_1^- the minus-spins cluster in the configuration σ_1 which is winding around the torus and by c_i^- all the other minus-spins clusters in σ_1. In σ_1 there is no direct interface $+ -$; then we can conclude that every c_i^- cluster is inside a zero-spins cluster. Now we consider the configuration σ_2 obtained from σ_1 by turning into zero all the minus spins in all the clusters c_i^-. The result $\sigma_1 \prec \sigma_2$ is obvious. We have also that $H(\sigma_2) \leqslant H(\sigma_1)$; indeed, in every cluster c_i^- there is at least one minus spin with two zero spins among its nearest neighbors; this spin can be transformed into zero, lowering the energy. We can repeat this argument until all the spins of the starting cluster c_i^- have been transformed into zero.

(iii) In σ_2 there is no direct interface $+ -$; then we observe that every cluster of plus spins is inside a cluster of zero spins; it can happen that in some of the plus-spins clusters there are one or more clusters of zero spins. We construct the configuration σ_3 by removing all these clusters of zero spins. With arguments similar to those used in step (ii), one can prove that $H(\sigma_3) \leqslant H(\sigma_2)$ and $\sigma_2 \prec \sigma_3$.

(iv) The configuration σ_3 is made of a minus-spins cluster which is winding around the torus, the zero-spins clusters denoted by c_i^0 $\forall i \in \{1, 2,..., k^0\}$, and the clusters with plus spins $c_{i,j}^+ \forall i \in \{1, 2,..., k^0\}$ and $\forall j \in \{1, 2,..., k_i^+\}$. The clusters $c_{i,j}^+ \forall j \in \{1, 2,..., k_i^+\}$ are all inside the cluster c_i^0. We consider now the rectangular envelopes $R_i^0 = R(c_i^0)$ $\forall i \in \{1, 2,..., k^0\}$ and the configuration σ_4 obtained by filling all these rectangles with zero spins; in this step the plus spins are not changed. It is immediate that $H(\sigma_4) \leqslant H(\sigma_3)$ and $\sigma_3 \prec \sigma_4$.

(v) Apart from the plus-spins cluster, the configuration σ_4 is made of zero rectangular clusters placed in the "sea" of minus spins. We obtain the configuration σ_5 by means of the *chain construction* used in ref. 9 applied to the rectangular clusters $R_i^0 \forall i \in \{1, 2,..., k^0\}$.

Let us briefly describe this construction. Given a set of rectangles $R_1^0,..., R_l^0$, we partition it into maximal connected components $\mathscr{C}_j^{(1)}$ with $j = 1,..., k^{(1)}$ called *chains of first generation*

$$(R_1^0 \cdots R_l^0) = (\mathscr{C}_1^{(1)} \cdots \mathscr{C}_{k^{(1)}}^{(1)})$$

The notion of connection is given by pairwise interaction: a set $R_1^0,..., R_m^0$ of rectangles is connected if it cannot be divided into two non-interacting parts.

Now consider the $k^{(1)}$ rectangles $R(\mathscr{C}_j^{(1)})$ obtained as the rectangular envelope of the union of the rectangles belonging to $\mathscr{C}_j^{(1)}$. Partition this set of rectangles into maximal connected components: in this way we construct the chains of second generation $\mathscr{C}_1^{(2)},..., \mathscr{C}_{k^{(2)}}^{(2)}$. We continue in this way up to a finite maximal order n such that the chains of the nth generation are noninteracting rectangles (see ref. 9 for more details).

We call σ_5 this configuration containing these noninteracting rectangular clusters $\bar{R}_i^0$ $\forall i \in \{1, 2,..., k^{0,f}\}$ of zero spins placed in the minus-spins "sea." With usual arguments one can prove that $H(\sigma_5) \leqslant H(\sigma_4)$ and $\sigma_4 \prec \sigma_5$.

(vi) By repeating the operations described in points (iv) and (v) for the plus-spins clusters lying in every rectangle $\bar{R}_i^0$ $\forall i \in \{1, 2,..., k^{0,f}\}$, we obtain the final configuration $\hat{\sigma}$. This configuration is made of the external rectangular zero-spins clusters $\bar{R}_i^0$ $\forall i \in \{1, 2,..., k^{0,f}\}$ and the internal noninteracting plus-spins clusters $\bar{R}_{i,j}^+ \forall i \in \{1, 2,..., k^{0,f}\}$ and $\forall j \in \{1, 2,..., k_i^{+,f}\}$. As usual, one can prove that $H(\hat{\sigma}) \leqslant H(\sigma_5)$ and $\sigma_5 \prec \hat{\sigma}$.

The definition of the map $\mathscr{F}$ is now complete; we observe that $\hat{\sigma}$ is a local minimum and that the properties (6.3) are satisfied. Finally, we remark that the map $\mathscr{F}$ is *monotone* in the sense that

$$\sigma \prec \eta \Rightarrow \hat{\sigma} \prec \hat{\eta} \tag{6.4}$$

for every pair of acceptable configurations σ and η.

Now we state the following result.

Proposition 6.1. We have

$$U(\mathscr{G}) \subset \hat{\mathscr{P}} \tag{6.5}$$

Namely, the set of minima of the energy in the boundary of $\mathscr{G}$ is contained in $\hat{\mathscr{P}}$.

Proof. In order to prove Proposition 6.1, we consider a configuration $\eta \in \partial\mathscr{G}$ and we show that there exists a special saddle $\tilde{\eta}$ such that $H(\eta) \geqslant H(\tilde{\eta})$. Then Proposition 6.1 will follow from Lemma 5.1.

Let us consider $\eta \in \mathscr{G}$; there exists a configuration $\sigma = \eta^{x,b}$ with $x \in \Lambda$ and $b \neq \eta(x)$ such that $\sigma \in \mathscr{G}$. By virtue of the monotonicity of the map $\mathscr{F}$ [see (6.4)] and of the fact that $\hat{\sigma}$ is a subcritical local minimum, it follows that $b < \eta(x)$; hence we also have that $b \neq +1$.

We denote by $R_i^0(\hat{\sigma})$ $\forall i \in \{1,..., k^0(\hat{\sigma})\}$ and by $R_{i,j}^+(\hat{\sigma})$ $\forall j \in \{1,..., k_i^+(\hat{\sigma})\}$ and $\forall i \in \{1,..., k^0(\hat{\sigma})\}$ the rectangles respectively of zeros and pluses which appear in the configuration $\hat{\sigma}$; we remark that all the rectangles $R_{i,j}^+(\hat{\sigma})$ $\forall j \in \{1,..., k_i^+(\hat{\sigma})\}$ are inside the zero rectangle $R_i^0(\hat{\sigma})$. In the following, by abuse of notation, we will also denote by $R_i^0(\hat{\sigma})$ what we will call the *structure* $R_i^0(\hat{\sigma})$, namely the complex given by the "external" rectangle together with all its "internal" rectangles of pluses (what before we called a plurirectangle is indeed a configuration containing a unique structure).

Case 1. $b = -1$ and $\eta(x) = 0$. From the definition of the map $\mathscr{F}$ it easily follows that necessarily x lies outside the rectangles $R_i^0(\hat{\sigma})$.

Given the configuration $\hat{\eta}$, we denote by $R_i^0(\hat{\eta})$ $\forall i \in \{1,..., k^0(\hat{\eta})\}$ and by $R_{i,j}^+(\hat{\eta})$ $\forall j \in \{1,..., k_i^+(\hat{\eta})\}$ and $\forall i \in \{1,..., k^0(\hat{\eta})\}$ the rectangles respectively of zeros and pluses which appear in it. We denote by $\bar{R}^0(\hat{\eta})$ the supercritical structure among the $R_i^0(\hat{\eta})$ and by $\bar{R}_1^0,..., \bar{R}_s^0$ the rectangles of zeros such that $\forall i \in \{1,..., s\}$ $\bar{R}_i^0$ appears in $\hat{\sigma}$ and $\forall i \in \{1,..., s\}$ $\bar{R}_i^0$ is "inside" the rectangle $\bar{R}^0(\hat{\eta})$.

We consider the configuration η_1 defined as follows: $\eta_1(x) = 0$; all the other spins are minus except for the zeros and the pluses of the structures $\bar{R}_i^0$ $\forall i \in \{1,..., s\}$. It can be easily proved that $H(\eta) \geqslant H(\eta_1)$. We distinguish two cases as follows.

Case 1.1. All the rectangles of pluses which appear in η_1 are subcritical.

We consider the configuration $\eta_{1.1}$ obtained from η_1 by changing into zeros all the plus spins. We remark that $H(\eta_1) \geqslant H(\eta_{1.1})$ because $\eta_{1.1}$ has been constructed by removing subcritical rectangles of pluses.

With an Ising-like argument (see, e.g., ref. 9) one can prove that $H(\eta_{1.1}) \geqslant H(\mathscr{P}_2)$. Hence in Case 1.1 we find a special saddle with energy lower than the starting configuration η.

Case 1.2. In η_1 there exists at least one supercritical rectangle of pluses.

We consider the configuration $\eta_{1.2}$ obtained from η_1 by removing in every structure $\bar{R}_i^0$ $\forall i \in \{1,..., s\}$ all the subcritical rectangles of pluses and by filling with pluses the rectangular envelope of the union of the supercritical rectangles of pluses. We remark that every structure $\bar{R}_i^0$ $\forall i \in \{1,..., s\}$ in $\eta_{1.2}$ is either "empty" (with no rectangle of pluses inside) or it has just a rectangle of pluses inside and this rectangle is supercritical.

We denote by Q the unit square centered at the site $x \in \Lambda$; we distinguish the two following cases:

Case 1.2.1. One of the structures $\bar{R}_i^0$ $\forall i \in \{1,..., s\}$ of $\eta_{1.2}$ (we denote it by $\bar{R}_{1.2.1}^0$) interacts with Q and the structure obtained by filling with zeros the rectangular envelope of $\bar{R}_{1.2.1}^0 \cup Q$ is supercritical.

Let us denote by $\eta_{1.2.1}$ the configuration obtained by removing in η_1 all the structures $\bar{R}_i^0$ $\forall i \in \{1,..., s\}$ except for $\bar{R}_{1.2.1}^0$. If Q is adjacent to $\bar{R}_{1.2.1}^0$, then $\eta_{1.2.1}$ is a special saddle. Otherwise Q is at distance one from one of the sides of the rectangle $\bar{R}_{1.2.1}^0$ or Q and $\bar{R}_{1.2.1}^0$ touch in a corner; in this case one can easily find a special saddle with energy lower than $H(\eta_{1.2})$.

Hence in Case 1.2.1 a special saddle with energy lower than the starting configuration η has been found.

Case 1.2.2. Condition 1.2.1 is not fulfilled.

By an argument similar to the one used in ref. 9 (see pp. 1136–1137 therein) we can find two structures $\tilde{R}_1$ and $\tilde{R}_2$ such that they are both subcritical, their external rectangles are interacting, the structure obtained by filling with zeros their rectangular envelope is supercritical, and $H(\eta_{1.2}) \geqslant H(\tilde{R}_1) + H(\tilde{R}_2)$ [when we say $H(\tilde{R}_i)$ with $i \in \{1, 2\}$ we are referring to the energy of the configuration obtained by plunging the structure $\tilde{R}_i$ into the "sea" of minuses]. We still have to distinguish between two possible cases.

Case 1.2.2.1. Both structures $\tilde{R}_i$ with $i \in \{1, 2\}$ have a supercritical rectangle of pluses inside.

Now we consider a just-supercritical frame whose external rectangle is contained in the rectangular envelope of the union of the two external rectangles of $\tilde{R}_1$ and of $\tilde{R}_2$. Such a frame surely exists and we denote it by $\tilde{C}$.

Starting from $\tilde{R}_1$ and $\tilde{R}_2$ and recalling that these structures are subcritical, one can construct two other structures, $\mathfrak{R}_1$ and $\mathfrak{R}_2$ (birectangles or frames), such that the three following conditions are satisfied: (i) $H(\tilde{R}_1) \geqslant H(\tilde{\mathfrak{R}}_1)$ and $H(\tilde{R}_2) \geqslant H(\tilde{\mathfrak{R}}_2)$; (ii) if the two external rectangles of the two structures $\tilde{\mathfrak{R}}_1$ and $\tilde{\mathfrak{R}}_2$ touch at a corner, then the rectangular envelope of the union of the external rectangles of $\tilde{\mathfrak{R}}_1$ and of $\tilde{\mathfrak{R}}_2$ coincides exactly with the external rectangle of the frame $\tilde{C}$; (iii) at least one of the two internal rectangles of pluses (the one in $\tilde{\mathfrak{R}}_1$ or the one in $\tilde{\mathfrak{R}}_2$) is supercritical.

If one considers the configuration $\eta_{1.2.2.1}$ obtained by plunging the structures into the "sea" of minus spins such that the external rectangles of $\tilde{\mathfrak{R}}_1$ and of $\tilde{\mathfrak{R}}_2$ touch at a corner, one can easily convince oneself that $H(\eta_{1.2}) \geqslant H(\eta_{1.2.2.1})$.

Finally, starting from $\eta_{1.2.2.1}$ we construct the special saddle $\tilde{\eta}$ by performing the following steps: (i) we fill with zeros the rectangular envelope of the union of the two external rectangles of zeros in $\eta_{1.2.2.1}$; (ii) we let grow the internal supercritical rectangle of pluses until the frame $\tilde{C}$ is reached; (iii) we transform into zeros all the pluses, except for one, of one of the four sides of the internal rectangle, such that a special saddle is obtained. It can be easily proved that $H(\eta_{1.2.2.1}) > H(\tilde{\eta})$ by comparing the energy differences involved in the three steps described above. We remark that the energy increase of the third step is largely compensated by the energy decrease involved in the second step.

Case 1.2.2.2. One of the structures $H(\tilde{R}_i)$ is "empty," in the sense that it has no rectangles of pluses inside.

This case can be discussed with arguments similar to those used in Case 1.2.2.1.

Case 2. $b = -1$ and $\eta(x) = +1$. Starting from $\hat{\sigma}$, one can always construct a configuration η_2 such that (i) $\eta_2 \in \partial\mathscr{G}$ and (ii) $\exists y \in \Lambda$ such that $\eta_2(y) = 0$ and $\eta_2^{y,-1} \in \mathscr{G}$. In this way the proof has been reduced to Case 1.

Case 3. $b = 0$. The site x is inside one of the rectangles of zeros $R_i^0(\hat{\sigma})\ \forall i \in \{1,..., k^0(\hat{\sigma})\}$; we denote it by $\bar{R}^0$. There are two possible cases that must be considered.

Case 3.1. x is not on one of the boundary slices of $\bar{R}^0$ (the typical situation is depicted in Fig. 42).

In this case the rectangles of zeros in $\hat{\eta}$ coincide with those in $\hat{\sigma}$, but the structure $\hat{R}^0(\hat{\eta})$ is supercritical [$\hat{R}^0(\hat{\eta})$ is the structure of $\hat{\eta}$ such that its external rectangle of zeros coincides with $\hat{R}^0$].

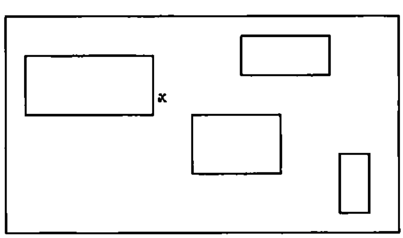


Fig. 42.

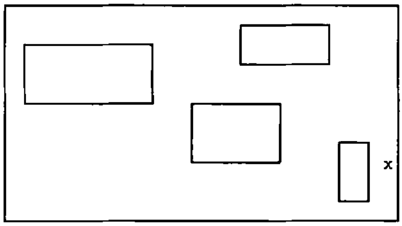


Fig. 43.

We consider the configuration $\eta_{3.1}$ defined as follows: (i) $\eta_{3.1}$ is obtained starting from $\hat{\sigma}$ by removing all the structures $R_i^0(\hat{\sigma})$ $\forall i \in \{1,..., k^0(\hat{\sigma})\}$ except for the one whose external rectangle coincides with the external rectangle of the structure $\bar{R}^0$ [we denote this structure by $\bar{R}^0(\hat{\sigma})$]; (ii) $\eta_{3.1}(x) = +1$. It can be easily proved that $H(\eta) \geqslant H(\eta_{3.1})$.

We denote by R_1 the rectangular envelope of the union of the supercritical rectangles of pluses inside $\bar{R}^0(\hat{\sigma})$ and by R_2 the rectangular envelope of the union of the supercritical rectangles of pluses inside $\bar{R}^0(\hat{\eta})$. We remark that the two structures $\hat{R}^0(\hat{\sigma})$ and $\bar{R}^0(\hat{\eta})$ have different internal rectangles of pluses, even though their external rectangles of zeros coincide.

Now we observe that there exists a rectangle R_3 contained in R_2 and containing R_1 such that the configuration with all the spins minus except for the zeros in the rectangle $\bar{R}^0$ and the pluses in R_3 is an almost-supercritical configuration. We consider the special saddle $\tilde{\eta}$ obtained by properly putting a unit plus protuberance to one of the four sides of the internal rectangle of pluses of the almost-supercritical configuration found before. It can be easily shown that $H(\eta_{3.1}) \geqslant H(\tilde{\eta})$. Hence, even in this case, we have found a special saddle with energy lower than the energy of the starting configuration $\eta \in \partial\mathscr{G}$.

Case 3.2. x is on one of the boundary slices of $\bar{R}^0$ (see, for example, Fig. 43).

We construct the configuration $\eta_{3.2}$ starting from $\hat{\sigma}$ and by turning into zero only the spin minus at a site nearest neighbor to x. One can easily convince oneself that $H(\hat{\eta}) \geqslant H(\eta_{3.2})$. If $\eta_{3.2} \in \partial\mathscr{G}$, then the proof is reduced to Case 1; if $\eta_{3.2} \in \mathscr{G}$, the proof is reduced to Case 3.1.

The proof of Proposition 6.1 is now complete. ∎

7. PROOF OF THE THEOREMS

Let us first give some definitions extending the ones given in Section 4. We recall that by $C(l_1, l_2)$ we denote the set of configurations containing only a frame with internal sides l_1, l_2. We recall the notation $l := \min\{l_1, l_2\}$, $m := \max\{l_1, l_2\}$.

We denote by $S(l_1, l_2)$ the set of configurations obtained from $C(l_1, l_2)$ by substituting one of the smaller internal sides with a unit square protuberance, namely by substituting all but one plus spin adjacent from the interior to one of the internal sides of length l with zeros (see Fig. 44).

We denote by $R(l_1, l_2)$ the set of configurations containing a unique birectangle obtained by erasing the internal unit-square protuberance from $S(l_1, l_2)$ (see Fig. 44). We denote by $G(l_1, l_2)$ the set of configurations obtained from the frame $C(l_1, l_2)$ by adding a unit-square spin-0 protuberance to one of the longer external sides in $C(l_1, l_2)$. A particularly relevant case will be the one $|l_1 - l_2| \leqslant 1$ where either $m = l + 1$ or $m = l$. We remark that $G(l-1, l)$ is obtained from the birectangle $R(l, l)$ by substituting one "free" external row (or column) of zeros of length $l + 2$ with a unit-square protuberance (see Fig. 44); similarly $G(l-1, l-1)$ is

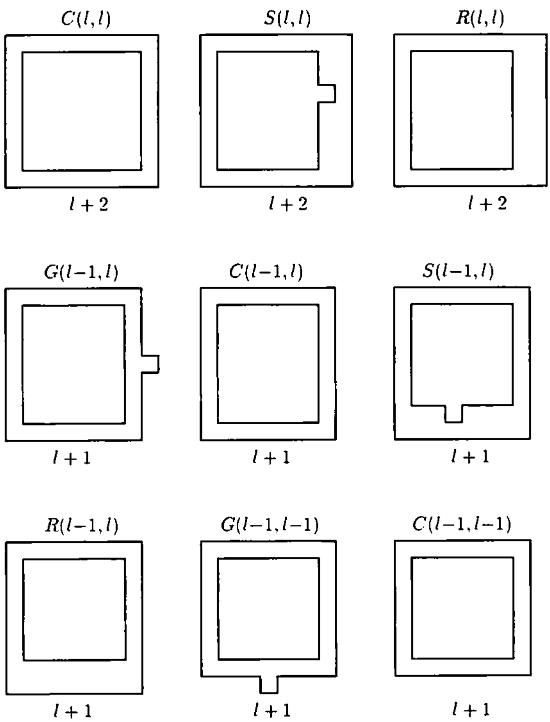


Fig. 44. Contraction of a squared frame. The energy differences involved in each single step of the contraction are $(h+\lambda)(l-1)$, $-[2J-(h+\lambda)]$, $(h-\lambda)(l+1)$, $-[2J-(h-\lambda)]$, $(h+\lambda)(l-2)$, $-[2J-(h+\lambda)]$, $(h-\lambda)\,l$, $-[2J-(h-\lambda)]$.

obtained from $R(l-1, l)$ by substituting one free external row or column of spin 0 of length $l+1$ with a unit-square protuberance.

Finally let us denote by $\bar{R}(l_1, l_2) := R(0, 0, l_1, l_2) \cup R(0, 0, l_2, l_1)$ the set of configurations without plus spins where the zero spins are precisely the ones contained inside a rectangle with sides equal, respectively, to l_1, l_2.

We want to prove now Theorem 1.

Let $\mathscr{P}$ be the set of protocritical saddles or special minimal saddles.

If $0 < 2\lambda < h$: $\mathscr{P} = \mathscr{P}_2$ in Fig. 32; namely $\mathscr{P}$ is the set of configurations with no pluses and a unique cluster of zeros given by a rectangle with sides M^*, $M^* - 1$ with a unit-square protuberance attached to one of its longer sides.

If $0 < \lambda < h < 2\lambda$ and $\delta < (h+\lambda)/2h$, then $\mathscr{P} = \mathscr{P}_{1,a} := S(l^*, l^*)$. If $0 < \lambda < h < 2\lambda$ and $\delta > (h+\lambda)/2h$, then $\mathscr{P} = \mathscr{P}_{1,b} := S(l^* - 1, l^*)$ (see Fig. 32).

Now we notice that the set $\mathscr{G} \subset \Omega_\Lambda$ defined in Section 6 satisfies the following properties:

- $\mathscr{G}$ connected; $-\underline{1} \in \mathscr{G}$, $+\underline{1} \notin \mathscr{G}$.
- There exists a path ω: $-\underline{1} \to \mathscr{P}$, contained in $\mathscr{G}$, with

$$H(\sigma) < H(\mathscr{P}) \qquad \forall \sigma \in \omega, \quad \sigma \neq \mathscr{P} \tag{7.1}$$

and there exists a path ω': $\mathscr{P} \to +\underline{1}$, contained in $\mathscr{G}^c$, with

$$H(\sigma) < H(\mathscr{P}) \qquad \forall \sigma \in \omega', \quad \sigma \neq \mathscr{P} \tag{7.2}$$

In the case $\mathscr{P} = \mathscr{P}_1$, (7.2) easily follows from the arguments of the proof of Proposition 4.1: ω is constructed following a sequence of shrinking subcritical droplets, whereas ω' follows a sequence of growing supercritical droplets. In the case $\mathscr{P} = \mathscr{P}_2$, (7.2) follows from the arguments of the proof of Proposition 4.2.

- The minimal energy in $\partial \mathscr{G}$ is attained only for "protocritical" (global saddle) configurations $\sigma \in \mathscr{P}$; namely,

$$\min_{\sigma \in \partial \mathscr{G}} (H(\sigma) - H(-\underline{1})) = H(\mathscr{P}) - H(-\underline{1}) =: \Gamma \tag{7.3}$$

$$\min_{\sigma \in \partial \mathscr{G} \notin \mathscr{P}} (H(\sigma) - H(\mathscr{P})) > 0 \tag{7.4}$$

We notice that, starting from any $\sigma \in \mathscr{P}$, we can change a spin adjacent to the unit-square protuberance always present in $\mathscr{P}$ (from -1 to 0 in $\mathscr{P}_2$ if $h > 2\lambda$ and from 0 to $+1$ in $\mathscr{P}_{1,a}$ or $\mathscr{P}_{1,b}$ if $h < 2\lambda$) in order to get a "stable protuberance of length 2." This protuberance is called stable since its growth takes place decreasing the energy while its shrinking requires an increase of energy. The probability off the above-described single spin change is not smaller than $1/|\Lambda|$ (see, for instance, ref. 13 for more details on this point).

In other words, with probability separated from zero, uniformly in β, starting from $\mathscr{P}$, we reach the strict basin of attraction of a supercritical minimum. Then, for any $\varepsilon > 0$, it follows from Proposition 4.1 that the probability to reach $+\underline{1}$ before reaching $-\underline{1}$ can be bounded from below as

$$P_{\mathscr{P}}(\tau_{+\underline{1}} < \tau_{-\underline{1}}) > \exp(-\varepsilon\beta) \tag{7.5}$$

We get from Proposition 4.1 that, for β sufficiently large, the typical time starting from $\mathscr{P}_1$ to reach $+\underline{1}$ is much shorter than the typical time to get to $\mathscr{P}$ starting from $-\underline{1}$,

$$\lim_{\beta \to \infty} P_{\mathscr{P}_1}(\tau_{+\underline{1}} < \exp(\Gamma_1) \mid \tau_{+\underline{1}} < \tau_{-\underline{1}}) = 1 \tag{7.6}$$

for a suitable $\Gamma_1 < \Gamma$.

Moreover, by an analysis totally analogous to the one needed for the Ising model (see, for instance, ref. 13) one can get the same results starting from $\mathscr{P}_2$; namely

$$\lim_{\beta \to \infty} P_{\mathscr{P}_2}(\tau_{+\underline{1}} < \exp(\Gamma_2) \mid \tau_{+\underline{1}} < \tau_{-\underline{1}}) = 1 \tag{7.7}$$

for a suitable $\Gamma_2 < \Gamma$.

In the appendix we state and prove a result concerning the sequence of passages through $\mathscr{P}$ and the typical time to see an "efficient" passage through $\mathscr{P}$, namely one followed by a descent to $+\underline{1}$.

From Propositions 3.4 and 3.7 in ref. 15, Proposition A.1 of the appendix, (7.5) and (7.6) we easily get Theorem 1. ∎

We want now to give the definition of the tube $\mathscr{T}$ of trajectories appearing in the statement of Theorem 2 below. It represents the typical mechanism of escape from metastability in the sense that, with probability tending to 1 as β tends to infinity, during its first excursion from $-\underline{1}$ to $+\underline{1}$, our process will follow a path in $\mathscr{T}$.

$\mathscr{T}$ will be optimal in the sense that it cannot be really reduced without losing in probability.

$\mathscr{T}$ involves a sequence of "droplets" with suitable geometric shapes and suitable "resistance times" in some "permanence sets" of configurations related to these droplets. The precise statement about the typical paths during the first excursion between $-\underline{1}$ and $+\underline{1}$ will involve a certain randomness of these resistance times inside the different permanence sets appearing in $\mathscr{T}$.

In $\mathscr{T}$ we will distinguish two parts: the "up" part $\mathscr{T}_u$, namely the ascent to $\mathscr{P}$, and the "down" part $\mathscr{T}_d$ from $\mathscr{P}$ to $+\underline{1}$. This second part $\mathscr{T}_d$ is *almost downhill* in the sense that all the paths $\omega = \sigma_0, \sigma_1,..., \sigma_i, \cdots \in \mathscr{T}_d$ will be such that

$$\sigma_0 = \mathscr{P}, \quad \exists \bar{T}\colon \sigma_{\bar{T}} = +\underline{1}, \qquad \max_{\sigma \in \omega \setminus \mathscr{P}} H(\sigma) < H(\mathscr{P}), \quad \min_{\sigma \in \omega} H(\sigma) = H(+\underline{1})$$

whereas $\mathscr{T}_u$ is *almost uphill* in the sense that all the paths $\omega = \sigma_0, \sigma_1,..., \sigma_i, \cdots \in \mathscr{T}_u$ will be such that

$$\sigma_0 = -\underline{1}, \quad \exists \bar{T}\colon \sigma_{\bar{T}} = \mathscr{P}, \quad \max_{\sigma \in \omega \setminus \mathscr{P}} H(\sigma) < H(\mathscr{P}), \quad \min_{\sigma \in \omega} H(\sigma) = H(-\underline{1})$$

In the following we give the definition of the time-reversed tube $\bar{\mathscr{T}}$ of $\mathscr{T}_u$.

$\bar{\mathscr{T}}$ will also be almost downhill; it will describe the typical first "descent" from the protocritical saddle to $-\underline{1}$. By general arguments based on reversibility,[19] we will deduce the desired results on the first excursion from $-\underline{1}$ to $\mathscr{P}$ saying that with probability tending to one as β tends to ∞ it takes place in the tube $\mathscr{T}_u$. Then to conclude our construction of $\mathscr{T}$ we will only have to determine $\mathscr{T}_d$.

Let us now recall some basic definitions of ref. 15 concerning the first descent from any configuration η_0 contained in a given cycle A to the bottom $F(A)$ valid not only for our Blume–Capel Metropolis dynamics, but also for a general "low-temperature" Markov chain satisfying Hypothesis M in the appendix. We refer to the appendix, where this more general setup is introduced.

We will first define in general the set of "standard cascades" emerging from a configuration η_0; our intention is to apply a (simplified version of a) result of ref. 15 stating that with high probability when $\beta \to \infty$ the first descent from η_0 to $F(A)$ follows, in a well-specified way, a standard cascade. Thus the main model-dependent work will be to determine, in our specific case, the set of standard cascades. In particular we will reduce the problem of the determination of the tube of typical trajectories followed by our process during its first descent to $-\underline{1}$ starting from a configuration σ_0 in $\mathscr{G}$ immediately reached starting from the global saddle $\mathscr{P}$, along a downhill path entering $\mathscr{G}$, to finding the set (denoted by $\hat{\mathscr{T}}$) of all the standard cascades (in a suitable cycle) emerging from σ_0.

A *standard cascade* emerging from a state η_0 is a sequence

$$\begin{aligned}\mathscr{T}(\eta_0, \omega_1, \eta_1, \omega_2,..., \eta_{M-1}, \omega_M) \\ = \omega_1 \cup Q_1 \cup \omega_2,..., Q_{M-1} \cup \omega_M \cup Q_M \qquad (7.8)\end{aligned}$$

where for $i = 1,..., M$: ω_i is a downhill path emerging from η_{i-1} and ending inside the "permanence set" Q^i. Each path ω_i can be downhill continued up to a stable equilibrium point $\xi_i \in Q_i$. The Q_i are special sets, being a sort of generalized cycle containing also the minimal saddles between ξ_i and $F(A)$; for $i = 1,..., M-1$, $\eta_i \in Q_i$ are minimal saddles between ξ_i and $F(A)$; finally, $\xi_M \subseteq Q_M \subseteq F(A)$ (see ref. 15, Section 4 for more details).

Notice that ω_i can just reduce to one downhill step from η_{i-1} to Q_i; in this case we use the convention $\omega_i = \eta_{i-1}$.

We do not give here the precise definition of the Q_i since it happens that we do not really need it. In our particular case of Metropolis dynamics for the Blume–Capel model with particular initial conditions (of interest for our applications) as we will check we have some simplifications w.r.t the general case.

The most important is that there Q_i for $i = 1,..., M-1$ are replaced by genuine cycles A_i; η_i, not contained in A_i, is an element of $\mathscr{S}(A_i)$ and ω_i ends in the interior of A_i.

We will apply the general theory developed in ref. 15 to two cases. In the first one, when analyzing $\bar{\mathscr{T}}$ the cycle A will be the maximal connected set $\bar{A}$ in Ω_Λ containing $-\underline{1}$ with energy less than $H(\mathscr{P})$. It follows from Proposition 3.4 in ref. 15 that $\bar{A}$ is contained in the set $\mathscr{G}$ introduced in Section 6 and that $\mathscr{S}(\bar{A}) \equiv \mathscr{P}$. Always in this case we haven $F(\bar{A}) \equiv Q_M \equiv \xi_M \equiv -\underline{1}$.

In the second case, in the study of $\mathscr{T}_d$ the cycle A will be the maximal connected component $\tilde{A}$ in Ω_Λ containing $+\underline{1}$ with energy less than $H(\mathscr{P})$. It is immediate to see that $\tilde{A} \subset \mathscr{G}^c$. In this case we have $F(\tilde{A}) \equiv Q_M \equiv \xi_M \equiv +\underline{1}$.

In both cases as we said before, for suitable initial conditions we will verify that the Q_i for $i = 1,..., M-1$ are replaced by genuine cycles A_i; M will depend on the initial configuration η_0 as well as on the particular choice of the parameters J, h, λ. The path ω_i ends in the interior of A_i; $\eta_i \in \mathscr{S}(\xi_i, F(A_{i+1}))$, not contained in A_i as we said before are minimal saddles in the boundary ∂A_i. The cycles A_i are precisely the maximal connected components containing ξ_i with energy less than $H(\eta_i)$ ($\xi_i \in A_i$ are the minima toward which ω_i can be downhill continued).

We consider an initial configuration η_0 corresponding to one of the following five cases:

1. $A = \bar{A}$: $\eta_0 \in \bar{A} \cap [R(l, l) \cup R(l, l+1)]$ for some $l \geqslant \tilde{L}$.
2. $A = \bar{A}$: $\eta_0 \in \bar{R}(M^*, M^*-1)$.
3. $A = \tilde{A}$, $0 < \lambda < h < 2\lambda$ and $\delta > (h+\lambda)/2h$: $\eta_0 \in \bar{B}(C(l^*-1, l^*))$.
4. $A = \tilde{A}$, $0 < \lambda < h < 2\lambda$ and $\delta < (h+\lambda)/2h$: $\eta_0 \in \bar{B}(C(l^*, l^*))$.
5. $A = \tilde{A}$: $\eta_0 \in \bar{B}(\bar{R}(M^*, M^*))$.

Remark. We could even consider much more general initial configurations η_0. It is not true (see the definition of the set $\bar{D}$ in Section 4) that for *any* η_0 the simplified version (involving genuine cycles A_i in place of the sets Q_i) of the general results of ref. 15 holds true. In fact with the very particular choice $\eta_0 \in \bar{A} \cap [R(l, l) \cup R(l, l+1)]$ as we will see an even simpler statement holds: the ω_i will be almost all coinciding with η_i (in the above-specified sense).

Warning. We want to warn the reader of the use that we are going to make, in the construction of the tube $\mathscr{T}$, of the equivalence class of configurations as it has been specified in the remark in the proof of Proposition 4.1. In fact, strictly speaking, what we will construct and call standard cascades are *sets* of standard cascades obtained from equivalence classes of configurations modulo rotations, translations, inversion, and "displacement of protuberances."

Let us now start with the definition of the set $\bar{\mathscr{T}}$ of the standard cascades emerging from a configuration σ_0 in $\mathscr{G}$ immediately reached starting from the global saddle $\mathscr{P}$ along a downhill path entering into $\mathscr{G}$.

We consider first the case $a = h/\lambda < 2$. The other case of $a > 2$ is almost identical to the corresponding one for the Ising model and will be treated later.

We have to distinguish two cases: $\delta = l^* - [2J - (h - \lambda)]/h < (h + \lambda)/2h$, when the global saddle $\mathscr{P}$ has the form $\mathscr{P}_{1,a} = S(l^*, l^*)$ given in Fig. 32; or $\delta > (h + \lambda)/2h$ when the global saddle has the form $\mathscr{P}_{1,b} = S(l^* - 1, l^*)$ also given in Fig. 32.

Let us first consider the case $\delta < (h + \lambda)/2h$ (as in Fig. 45 for $l = l^*$). Let $\bar{\mathscr{P}}_1 = R(l^*, l^*)$ be the configuration obtained from $\mathscr{P}_1$ by erasing the unit-square protuberance. $\bar{\mathscr{P}}_1$ is a subcritical birectangle; it belongs to the set $\mathscr{G}$ and satisfies condition 1 above.

To construct the tube $\bar{\mathscr{T}}$ we have basically to solve the above-described sequence of minimax problems. To simplify the exposition we divide the tube $\bar{\mathscr{T}}$ into four segments corresponding to four different mechanisms of contraction; we write

$$\bar{\mathscr{T}} = \bar{\mathscr{T}}_1 \cup \bar{\mathscr{T}}_2 \cup \bar{\mathscr{T}}_3 \cup \bar{\mathscr{T}}_4 \tag{7.9}$$

The most relevant ones are the first and the second parts. As we will see, the third part for $h < 2\lambda$ reduces just to a simple downhill path.

We start from the determination of the minimal saddle $\eta_1 := \mathscr{S}(\bar{\mathscr{P}}_1, -\underline{1})$ between $\bar{\mathscr{P}}_1$ and $-\underline{1}$.

From the results of Section 4 we know that $\mathscr{S}(\bar{\mathscr{P}}_1, -\underline{1})$ is not trivial, in the sense that it differs from $\bar{\mathscr{P}}_1$ and

$$\eta_1 := \mathscr{S}(\bar{\mathscr{P}}_1, -\underline{1}) = S(l^* - 1, l^*) \tag{7.10}$$

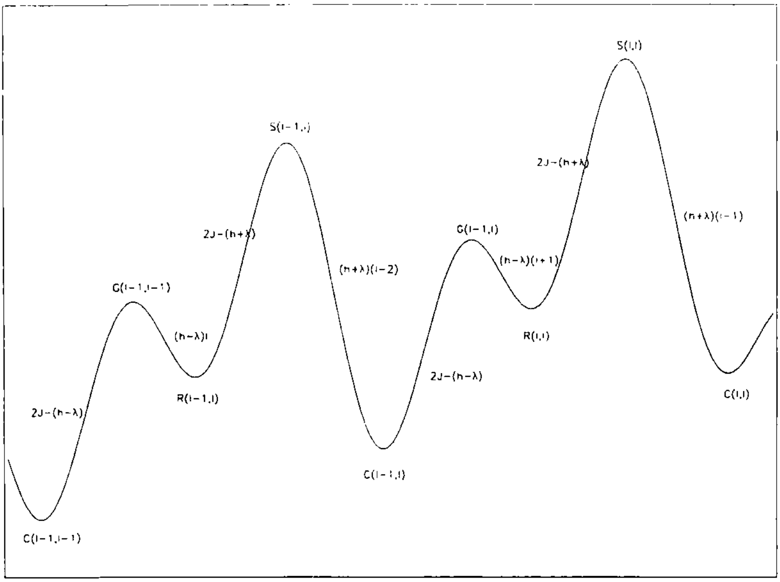


Fig. 45.

Thus the first "permanence set" Q_1 of our standard cascade is the cycle A_{l^*-1,l^*} defined as the maximal connected set of configurations containing $R(l^*, l^*)$ with energy less than $H(S(l^*-1, l^*))$. We recall that the basic inequality to be checked in order to get (7.10) is

$$H(S(l, l)) - H(S(l-1, l)) > 0$$

which is satisfied for $L^* \leqslant l \leqslant l^* - 1$.

For any l: $L^* \leqslant l \leqslant l^* - 1$ we define the cycle $A_{l,l}(A_{l,l+1})$ as the maximal connected set of configurations containing $R(l, l)[R(l, l+1)]$ with energy less than $H(S(l-1, l))[H(S(l, l))]$ (see Fig. 45). By extending the previous argument we get that the first part of our standard cascade is given by

$$\begin{aligned}\mathcal{T}_1 = A_{l^*-1,l^*}, S(l^*-1, l^*), A_{l^*-1,l^*-1}, S(l^*-1, l^*-1),\\ A_{l^*-2,l^*-1}, \ldots, S(\tilde{L}, \tilde{L}+1), A_{\tilde{L},\tilde{L}}, S(\tilde{L}, \tilde{L})\end{aligned} \tag{7.11}$$

Then we observe that for $l < \tilde{L} - 1$, we have

$$H(S(l, l)) < H(G(l, l)) \tag{7.12}$$

$$H(S(l, l+1)) < H(G(l, l+1))$$

(7.12) are the basic inequalities to get that, for $l_0 \leqslant l < \tilde{L}$,

$$\begin{aligned} \mathscr{S}(R(l, l+1), -\underline{1}) &= G(l, l) \\ \mathscr{S}(R(l, l), -\underline{1}) &= G(l-1, l) \end{aligned} \tag{7.13}$$

It is clear from the results of Section 4 that the subsequent permanence sets are the cycles

$$\begin{aligned} &A^1_{\tilde{L}, \tilde{L}-1}, A^2_{\tilde{L}, \tilde{L}-1}, A^1_{\tilde{L}-1, \tilde{L}-1}, A^2_{\tilde{L}-1, \tilde{L}-1}, \ldots, A^1_{l, l}, A^2_{l, l}, \\ &A^1_{l, l-1}, A^2_{l, l-1}, \ldots, A^2_{l_0, l_0} \end{aligned} \tag{7.14}$$

where $l_0 = [h/\lambda + 1]$, $l_0 \leqslant l$; we notice that for our present choice of the parameters $\lambda < h < 2\lambda$, we have $l_0 = 2$, but we could consider a general situation $l_0 > 2$ as well when analyzing the contraction of a subcritical frame in the region $h > 2\lambda$ (case 1 above). Moreover, for $l_0 \leqslant l \leqslant \tilde{L}$:

$A^1_{l, l-1} =$ maximal connected component of the set of configurations containing $R(l, l)$ with energy less than $H(G(l-1, l))$; namely $A^1_{l, l-1}$ is the strict basin of attraction of $R(l, l)$:

$$A^1_{l, l-1} = \bar{B}(R(l, l))$$

with bottom

$$F(A^1_{l, l-1}) = R(l, l)$$

and minimal saddle

$$\mathscr{S}(A^1_{l, l-1}) = G(l-1, l)$$

$A^2_{l, l-1} =$ maximal connected component containing $C(l-1, l)$ with energy less than $H(S(l-1, l))$. We have

$$A^2_{l, l-1} = \bar{B}(C(l-1, l))$$

$$F(A^2_{l, l-1}) = C(l-1, l)$$

and

$$\mathscr{S}(A^2_{l, l-1}) = S(l-1, l)$$

For $l_0 + 1 < l \leqslant \tilde{L}$ we define:

$A^1_{l-1, l-1} =$ maximal connected component containing $R(l-1, l)$ with energy less than $H(G(l-1, l-1))$. We have

$$A^1_{l-1, l-1} = \bar{B}(R(l-1, l))$$

$$F(A^1_{l-1, l-1}) = R(l-1, l)$$

and

$$\mathscr{S}(A^1_{l-1,l-1}) = G(l-1, l-1)$$

$A^2_{l-1,l-1}$ = maximal connected component containing $C(l-1, l-1)$ with energy less than $H(S(l-1, l-1))$. We have

$$A^2_{l-1,l-1} = \bar{B}(C(l-1, l-1))$$

$$F(A^2_{l-1,l-1}) = C(l-1, l-1)$$

and

$$\mathscr{S}(A^2_{l-1,l-1}) = S(l-1, l-1)$$

Then the second segment of the standard cascade is

$$A^1_{\tilde{L},\tilde{L}-1}, G(\tilde{L}-1, \tilde{L}), A^2_{\tilde{L},\tilde{L}-1}, S(\tilde{L}-1, \tilde{L}), A^1_{\tilde{L}-1,\tilde{L}-1}, G(\tilde{L}-1, \tilde{L}-1),$$
$$A^2_{\tilde{L}-1,\tilde{L}-1}, S(\tilde{L}-1, \tilde{L}-1),..., S(l_0, l_0), A^1_{l_0,l_0-1} \tag{7.15}$$

We notice that both the first and the second parts $\bar{\bar{\mathscr{T}}}_1$, $\bar{\bar{\mathscr{T}}}_2$ of the tube $\bar{\bar{\mathscr{T}}}$ describe a contraction following squared or almost squared frames; but whereas in the first part the permanence sets are cycles with many minima in their interior, in the second part they are "one well" in the sense that they coincide with the strict basin of attraction of their bottoms. The typical times spent inside these cycles and the typical states visited before leaving them are different in the two cases of $\bar{\bar{\mathscr{T}}}_1$ and $\bar{\bar{\mathscr{T}}}_2$.

The third part $\bar{\bar{\mathscr{T}}}_3$, which we are going to define, corresponds to the shrinking of the interior rectangle of the frame $C(l_0, l_0)$. Indeed it follows from Section 4 [see (4.6) therein] that for $l < l_0$ the lowest minimal saddles in the boundary of the basin of attraction $A^1_{l_0,l_0-1}$ of the birectangle

$$R(l_0, l_0) \equiv R(l_0, l_0-1; l_0+2, l_0+2) \cup R(l_0-1, l_0; l_0+2, l_0+2)$$

is not $G(l_0-1, l_0)$ corresponding to S_2 in Fig. 20, but, rather, the saddle S_3 in Fig. 20; in other words, starting from the birectangle $R(l_0, l_0)$, it is no longer convenient to continue the contraction along frame shapes, but, on the contrary, the internal rectangle starts its independent shrinking, keeping the external rectangle fixed. It appears clear that if $h < 2\lambda$, then the shrinking and disappearing of the internal two-by-two rectangle is just a downhill path where the number of internal plus spins decreases monotonically to zero. If we were considering a general initial condition corresponding to the above case 1, namely the contraction of a subcritical frame for $h > 2\lambda$, then we would have had $l_0 > 2$ and the shrinking and disappearance of the internal rectangle would have followed a sequence of squared or almost-squared rectangular shape exactly as in the case of the standard Ising model.

In the following we will consider birectangles $R(L_1, L_2; M_1, M_2)$ [see (3.3)] also for $L_1, L_2 = 0, 1$.

Then the third part for $h < 2\lambda$ is just the downhill path:

$$\bar{\mathscr{T}}_3 = S_3^*, R(1, 2; 4, 4), R(1, 1; 4, 4), R(0, 0; 4, 4) \tag{7.16}$$

where by S_3^* we denote the saddle depicted in Fig. 20 when the external rectangle is a 4×4 square and the internal cluster is a "triangle made by three sites."

Finally, the fourth part is just an Ising-like contraction of the remaining 4×4 rectangle of zeros. We will observe first a sequence of permanence sets (corresponding to stable rectangles) and saddles and finally the downhill path describing the disappearance of the last 2×2 stable rectangle.

We have

$$\bar{\mathscr{T}}_4 = \tilde{S}_1, \tilde{R}_1, \tilde{S}_2, \tilde{R}_2, \tilde{S}_3, \tilde{R}_3, \tilde{S}_4, \tilde{R}_4, \tilde{\omega} \tag{7.17}$$

where $\tilde{R}_1 = R(0, 0; 4, 3) \cup R(0, 0; 3, 4)$, $\tilde{R}_2 = R(0, 0; 3, 3)$, $\tilde{R}_3 = R(0, 0; 3, 2) \cup R(0, 0; 2, 3)$, and $\tilde{R}_4 = R(0, 0; 2, 2)$; the downhill path $\tilde{\omega}$ is given by

$$\tilde{\omega} := \tilde{S}_5, \tilde{R}_5, \tilde{R}_6, -\underline{1} \tag{7.18}$$

with $\tilde{R}_5 = R(0, 0; 1, 2) \cup R(0, 0; 2, 1)$, $\tilde{R}_6 = R(0, 0; 1, 1)$; the saddles $\tilde{S}_i$, $i = 1,..., 5$, are obtained from the rectangles in $\tilde{R}_i$ by adding a unit-square protuberance to one of its longer sides.

This concludes the definition of $\bar{\mathscr{T}}$ for $h < 2\lambda$, $\delta < (h + \lambda)/2h$.

In the case $h < 2\lambda$, $\delta > (h + \lambda)/2h$ the definition of $\bar{\mathscr{T}}$ is almost identical; we only have to modify slightly at the very beginning the definition of $\bar{\mathscr{T}}_1$ by eliminating its first permanence set.

Indeed we know from Section 5 that now $H(S(l^* - 1, l^*)) > H(S(l^*, l^*))$, so that the protocritical saddle is, in this case $\mathscr{P}_{1, b} = S(l^* - 1, l^*)$. Now the configuration $\bar{\mathscr{P}}_1$ obtained from $\mathscr{P}_1$ by erasing the unit-square protuberance is the subcritical rectangle $\bar{\mathscr{P}}_1 = R(l^* - 1, l^*)$; again this belongs to case 1. Then the first permanence set is now $A_{l^* - 1, l^* - 1}$ and we have

$$\begin{aligned} \bar{\mathscr{T}}_1 = A_{l^* - 1, l^* - 1}^{\bullet}, S(l^* - 1, l^* - 1), A_{l^* - 2, l^* - 1} \cdots S(\tilde{L}, \tilde{L} + 1), \\ A_{\tilde{L}, \tilde{L}}, S(\tilde{L}, \tilde{L}) \end{aligned} \tag{7.19}$$

The other segments of the tube $\bar{\mathscr{T}}_i$, $i = 2, 3, 4$, are defined exactly as before.

The last case that we still have to analyze to construct $\bar{\mathscr{T}}$ is $h > 2\lambda$. In this case the protocritical saddle is $\mathscr{P} = \mathscr{P}_2$ and the tube $\bar{\mathscr{T}}$ is just an Ising-like contraction along squared or almost-squared rectangular clusters of zeros in a sea of minuses.

Now the configurations obtained by erasing the unit-square protuberance containing a unique subcritical rectangle of zeros in a sea of minuses is given by

$$\bar{\mathscr{P}}_2 = \bar{R}(M^* - 1, M^*)$$

Notice that $\mathscr{P}_2$ is included in case 2 above.

We observe that the appearance of a single plus spin will induce the overcoming of an energy barrier greater than or equal to $4J - (h + \lambda)$. It is very easy to see that we can proceeded with the construction of the set of the standard cascades emerging from $\bar{R}(M^* - 1, M^*)$ without being forced to overcome a barrier larger then $2J$, so that certainly in all these standard cascades, for our choice of the parameters, we will never see a single plus spin appearing. Indeed one can easily convince oneself that the sequence of minimax problems to be solved is the exact analog of the that arising in the analysis of a subcritical contraction for a standard Ising model. We refer to refs. 9 and 19 for more details. For completeness in the following we summarize the results using our notation.

The first permanence set is $\bar{B}(R(M^* - 1, M^*))$.

Let us define the following sequences of pairs of integers:

$$(l_1, m_1), (l_2, m_2),..., (l_N, m_N), \qquad N = 2M^* - 2$$

$$(l_1, m_1) = (M^* - 1, M^*), \qquad (l_N, m_N) = (1, 1)$$

$$|l_i - m_i| \leqslant 1: \qquad m_i = l_i \quad \text{or} \quad m_i = l_i + 1$$

$$\text{if} \quad (l_i, m_i) = (l, l+1) \qquad \text{then} \quad (l_{i+1}, m_{i+1}) = (l, l)$$

$$\text{if} \quad (l_i, m_i) = (l, l) \qquad \text{then} \quad (l_{i+1}, m_{i+1}) = (l-1, l)$$

Given (l, m) as before: for $|l - m| \leqslant 1$, $1 \leqslant m \leqslant M^* - 1$, we denote by $\bar{S}(l, m)$ the saddle obtained from $\bar{R}(l, m)$ by adding a unit-square protuberance (with a zero spin inside) to one of its longest sides.

We have

$$\bar{\mathscr{T}} = \bar{B}(\bar{R}(l_1, m_1)), \bar{S}(l_2, m_2), \bar{B}(\bar{R}(l_2, m_2)),..., \bar{S}(l_N, m_N), \bar{R}(l_N, m_N), \underline{-1}$$

This concludes the definition of $\bar{\mathscr{T}}$.

Let us now pass to the definition of the descent part $\mathscr{T}_d$ of the tube $\mathscr{T}$.

We start from the case $h < 2\lambda$, $\delta > (h + \lambda)/2h$.

It is immediately seen that by adding to $\mathscr{P}_1 = \mathscr{P}_{1,b} \equiv S(l^* - 1, l^*)$ a unit-square protuberance to form a stable protuberance of length 2 we get a configuration η_0 included in case 3.

We distinguish in $\mathscr{T}_d$ two parts: $\mathscr{T}_{d,1}$ and $\mathscr{T}_{d,2}$.

For $l^*-1 \leqslant l < \tilde{M}-2$ we denote by $\bar{A}_{l-1,l}$ the cycle given by the maximal connected set of configurations containing $C(l-1, l)$ with energy less than $H(S(l, l))$. We easily verify that $F(\bar{A}_{l-1,l}) = C(l-1, l)$, $\mathscr{S}(\bar{A}_{l-1,l}) = S(l-1, l)$.

For $l^* \leqslant l < \tilde{M}-2$ we denote by $\bar{A}_{l,l}$ the cycle given by the maximal connected set of configurations containing $C(l, l)$ with energy less than $H(S(l, l+1))$. We easily get that $F(\bar{A}_{l,l}) = C(l, l)$, $\mathscr{S}(\bar{A}_{l,l}) = S(l, l+1)$.

For $l^*-1 \leqslant l < \tilde{M}-2$ we denote by $\Omega_{l-1,l}$ the set of downhill paths starting from $S(l-1, l)$ and ending in $\bar{A}_{l-1,l}$. Similarly, for $l^* \leqslant l < \tilde{M}-2$ we denote by $\Omega_{l,l}$ the set of downhill paths starting from $S(l, l)$ and ending in $\bar{A}_{l,l}$. We set

$$\begin{aligned}\mathscr{T}_{d,1} = \eta_0, \bar{A}_{l^*-1,l^*}, S(l^*, l^*), \Omega_{l^*,l^*}, \bar{A}_{l^*,l^*},\\ S(l^*, l^*+1), \Omega_{l^*,l^*+1},..., S(\tilde{M}-2, \tilde{M}-2)\end{aligned}$$

As it has been shown in Section 4, for $l \geqslant \tilde{M}-2$ the growth is typically symmetric in the sense that the probability of growth in the directions parallel or orthogonal to the shortest side of our supercritical frame are logarithmically equivalent for large β. Moreover, it follows from the analysis developed in Section 4 that for $l \geqslant \tilde{M}-2$ the set $\mathscr{D}$ defined in (4.14) does not play any particular role and the permanence sets are cycles given by the strict basins of attraction of frames $C(l_1, l_2)$ or birectangles $R(l_1, l_2)$. The second part $\mathscr{T}_{d,2}$ of $\mathscr{T}_d$ will describe the supercritical growth starting from $l = \tilde{M}-2$. To construct $\mathscr{T}_{d,2}$ we need some more geometrical definitions.

For a given frame $C(l_1 l_2)$, we use the notation $C(l, m)$ to make explicit the shorter and longer sides l and m, respectively.

We denote by $G_>(l, m)$, $G_<(l, m)$, respectively, the saddle configurations containing a unique droplet obtained by attaching a unit-square protuberance (with a zero spin inside) to a longer or shorter external side of $C(l, m)$.

We denote by $R_>(l, m)$, $R_<(l, m)$, respectively, the birectangles obtained from $G_>(l, m)$, $G_<(l, m)$ by extending the unit-square protuberance to an entire side.

We denote by $S_>(l, m)$, $S_<(l, m)$, respectively, the saddle configurations containing a unique droplet obtained from $R_>(l, m)$, $R_<(l, m)$ by attaching a unit-square protuberance (with a plus spin inside) to the internal free side.

We denote by $\Omega_>(l, m)$, $\Omega_<(l, m)$, respectively, the set of all the downhill paths emerging from $S_>(l, m)$, $S_<(l, m)$ and ending in $\bar{B}(C(l+1, m))$, $\bar{B}(C(l, m+1))$; finally we denote by $\hat{\Omega}_>(l, m)$, $\hat{\Omega}_<(l, m)$ the set of all downhill paths emerging from $G_>(l, m)$, $G_<(l, m)$ and ending in $\bar{B}(R_>(l, m))$, $\bar{B}(R_<(l, m))$.

Given (l, m), we denote by $\Gamma_{>}(l, m)$ the sequence $\bar{B}(C(l, m))$, $G_{>}(l, m)$, $\hat{\Omega}_{>}(l, m)$, $\bar{B}(R_{>}(l, m))$, $S_{>}(l, m)$, $\Omega_{>}(l, m)$, $\bar{B}(C(l, m+1))$. Similarly we denote by $\Gamma_{<}(l, m)$ the sequence $\bar{B}(C(l, m))$, $G_{<}(l, m)$, $\hat{\Omega}_{<}(l, m)$, $\bar{B}(R_{<}(l, m))$, $S_{<}(l, m)$, $\Omega_{<}(l, m)$, $\bar{B}(C(l+1, m))$.

A sequence $(l_i, m_i)_{i=1,2,\ldots}$ with $l_i \leqslant m_i$ is called *regularly increasing* if: $l_1 = m_1 = \tilde{M} - 2$ and, for any $i = 1, 2,\ldots$, either $(l_{i+1}, m_{i+1}) \equiv (l_i, m_i)^{>} := (l_i + 1, m_i)$ or $(l_{i+1}, m_{i+1}) \equiv (l_i, m_i)^{<} := (l_i, m_i + 1)$.

If $l_i = m_i = L \equiv$ the side of our torus Λ, we set $l_{i+1} = m_{i+1} = L$.

Let $\mathscr{L}$ be the set of all regularly increasing sequences. For any $(l_i, m_i)_{i=1,2,\ldots} \in \mathscr{L}$ we define $\delta(l_i, m_i) := >$ if $(l_{i+1}, m_{i+1}) \equiv (l_i, m_i)^{>}$ and $\delta(l_i, m_i) := <$ if $(l_{i+1}, m_{i+1}) \equiv (l_i, m_i)^{<}$.

From the arguments developed in Section 4 it easily follows that the second part of $\mathscr{T}_d$ is given by

$$\mathscr{T}_{d,2} = \bigcup_{(l_i, m_i)_{i=1,2,\ldots} \in \mathscr{L}} \Gamma_{\delta(l_1, m_1)}(l_1, m_1)$$

$$\cup\, \Gamma_{\delta(l_2, m_2)}(l_2, m_2) \cup \cdots \cup \Gamma_{\delta(l_i, m_i)}(l_i, m_i) \cup \ldots$$

This concludes the construction of $\mathscr{T}_d$ for the case $h < 2\lambda$, $\delta > (h + \lambda)/2h$.

The case $h < 2\lambda$, $\delta < (h + \lambda)/2h$ requires only minor changes: the only difference is that now we have to start a step further. Indeed it is immediately seen that by adding to $\mathscr{P}_1 = \mathscr{P}_{1,a} \equiv S(l^*, l^*)$ a unit-square protuberance to form a stable protuberance of length 2 we get a configuration η_0 included in case 4. We have

$$\mathscr{T}_{d,1} = \eta_0, \bar{A}_{l^*, l^*}, S(l^*, l^* + 1), \Omega_{l^*, l^*+1}, \ldots, S(M^* - 2, M^* - 2).$$

The rest is identical.

For the case $h > 2\lambda$ we have exactly the same behavior as in the Ising model, namely we consider an initial condition as in case 5. Then we have a symmetric growth along a sequence of supercritical growing rectangles of zeros in a sea of minuses up to the configuration $\underline{0}$. Subsequently we have again an Ising-like nucleation of a protocritical droplet of pluses in the sea of zeros [an $L^* \times (L^* - 1)$ rectangle with a unit-square protuberance attached to one of its longer sides] up to the configuration $+\underline{1}$. This last case has been already analyzed in detail (see, for instance, refs. 13 and 19). We leave the details to the reader.

One can easily convince oneself that this indeed concludes the construction of the set of all standard cascades emerging from any of the above-specified five types of initial conditions for any value of the parameters (not only for the subcases that we have explicitly treated).

We can now state our main result on the tube of typical trajectories during the first excursion between $-\underline{1}$ and $+\underline{1}$.

Let $\mathscr{T} := \mathscr{T}_u \cup \mathscr{P} \cup \mathscr{T}_d$ with $\mathscr{T}_u$ given by the time reversal of the set of standard cascades in $\bar{A} \subset \mathscr{G}$ emerging from $\bar{\mathscr{P}}$: $\mathscr{T}_u := \mathscr{R}\bar{\mathscr{T}}$ (the time-reversal operator acts on paths in this way: for $\omega = (x_1, x_2,..., x_{N-1}, x_N)$: $\mathscr{R}\omega = (x_N, x_{N-1},..., x_2, x_1)$]; $\mathscr{T}_d$ given by the set of standard cascades in $\tilde{A} \subset \mathscr{G}^c$ emerging from $\tilde{\mathscr{P}}$. Let $\tilde{\mathscr{P}}_1$ be either $\tilde{\mathscr{P}}_1$ or $\tilde{\mathscr{P}}_2$ according to the values of the parameters J, h, λ; let $\bar{\mathscr{P}}_1$ be either $\bar{\mathscr{P}}_1$ or $\bar{\mathscr{P}}_2$ according to the values of the parameters J, h, λ.

Theorem 2. Consider the dynamical Blume–Capel model described by the Markov chain with transition probabilities given in (2.6) of Section 2. For any choice of the parameters J, h, λ compatible with (3.17) we have the following: (i)

$$\lim_{\beta \to \infty} P_{-\underline{1}}(\sigma_t \in \mathscr{T} \; \forall t \in [\bar{\tau}_{-\underline{1}}, \tau_{+\underline{1}}]) = 1$$

The history of the process in $\mathscr{T}$ is described in the following way: consider an initial configuration η_0 corresponding to one of the following five cases:

1. $A = \bar{A}$: $\eta_0 \in \bar{A} \cap [R(l, l) \cup R(l, l+1)]$ for some $l \geqslant \tilde{L}$.
2. $A = \bar{A}$: $\eta_0 \in \bar{R}(M^*, M^* - 1)$.
3. $A = \tilde{A}$, $0 < \lambda < h < 2\lambda$, and $\delta > (h + \lambda)/2h$: $\eta_0 \in \bar{B}(C(l^* - 1, l^*))$.
4. $A = \tilde{A}$, $0 < \lambda < h < 2\lambda$, and $\delta < (h + \lambda)/2h$: $\eta_0 \in \bar{B}(C(l^*, l^*))$.
5. $A = \tilde{A}$: $\eta_0 \in \bar{B}(\bar{R}(M^*, M^*))$.

Then, considering for any such A, η_0 the set of all standard cascades emerging from η_0 and falling into $F(A)$ we have (ii)

$$\exists \delta > 0 \qquad \text{such that} \quad \lim_{\beta \to \infty} P_{\eta_0}(\tau_F(A) < \exp(\beta[H(\eta_1) - H(F(A)) - \delta]) = 1$$

and (iii)

$$\lim_{\beta \to \infty} P_{\eta_0}(\forall t \leqslant \tau_{F(A)}: x_t \in \mathscr{T}(\eta_0, \omega_1, \eta_1, \omega_2,..., \eta_{M-1}, \omega_M)$$
$$\text{for some standard cascade } \eta_0, \omega_1, \eta_1, \omega_2,..., \eta_{M-1}, \omega_M) = 1,$$

and, moreover (iv) with probability $\to 1$ as $\beta \to \infty$, there exists a sequence $\eta_0, \omega_1, \eta_1, \omega_2,..., \eta_{M-1}, \omega_M$ such that our process starting at $t = 0$ from η_0, between $t = 0$ and $t = \tau_{F(A)}$, after having followed the initial downhill path ω_1, visits, sequentially, the sets $A_1, A_2,..., A_{M-1}$ exiting from A_j through η_j and then following the path ω_{j+1} before entering A_{j+1}.

For every $\varepsilon > 0$ with probability tending to one as $\beta \to \infty$ the process spends inside each A_j a time $T_j(\varepsilon)$:

$$\exp(\beta[H(\eta_j) - H(F(A_j)) - \varepsilon]) < T_j(\varepsilon) < \exp(\beta[H(\eta_j) - H(F(A_j)) + \varepsilon])$$

and before exiting from A_j it visits each point in A_j at least $\exp(\beta\varepsilon)$ times.

Proof. We easily get that

$$\lim_{\beta \to \infty} P_{\mathscr{P}_1}(\sigma_1 \in \bar{\mathscr{P}}_1 \mid \sigma_1 \in \mathscr{G}) = 1 \tag{7.20}$$

Indeed (7.20) follows from the fact that there is only one first possible step in any downhill path from $\mathscr{P}_1$ to $\mathscr{G}$: it corresponds to erasing the unit-square protuberance to get $\bar{\mathscr{P}}_1$. On the other hand, we have

$$\lim_{\beta \to \infty} P_{\mathscr{P}_{1,h}}(\sigma_1 = \eta_0 \mid \sigma_1 \in \mathscr{G}^c) = 1 \tag{7.21}$$

The proof is an immediate consequence of Theorem 1, (7.20), (7.21), Theorem 1 in ref. 15, and the results of ref. 20. ∎

8. CONCLUSIONS

We have described the metastable behavior of a dynamical Blume–Capel model. Our updating rule is given by the classical Metropolis algorithm, but it is clear that our results extend to a wide class of single-spin-flip reversible dynamics.

Our results refer to the asymptotic regime of small but fixed magnetic field h and chemical potential λ, large but fixed volume Λ, and very large inverse temperature β.

We take mainly the point of view of the so-called *pathwise approach* to metastability, aiming to describe the typical behavior of the random trajectories of our stochastic dynamics rather than describing the evolution of the averages.

The Blume–Capel model exhibits the interesting feature of the presence of three possible phases. The equilibrium phase diagram is, consequently, very rich and interesting. The most important aspect from the point of view of the study of the dynamics of metastability is the presence, near the triple point, of two competing metastable phases. This means that, for instance, if one wants to describe the decay from the metastable $-\underline{1}$ phase to the stable $+\underline{1}$ phase one has to take into account the presence of another metastable phase: $\underline{0}$.

We took as initial condition the state $-\underline{1}$ and we analyzed the region of parameters $0<\lambda<h$. Let us subdivide it into the regions II and III defined as follows:

$$\text{II} := 0<\lambda<h<2\lambda$$

$$\text{III} := 0<2\lambda<h$$

It is easily seen that, with the same arguments developed in Sections 3–6, we could analyze the region

$$\text{IV} := 0< -\lambda<h$$

as well. In regions II, III, and IV the stable equilibrium phase (absolute minimum for the energy) is $+\underline{1}$ and we have:

$$H(-\underline{1})>H(\underline{0})>H(+\underline{1})$$

In the region

$$I := 0<h<\lambda$$

we have

$$H(\underline{0})>H(-\underline{1})>H(+\underline{1})$$

and then it is reasonable to expect and not difficult to prove that in the decay from $-\underline{1}$ to $+\underline{1}$ the state $\underline{0}$ does not play any role. Indeed it is sufficient to exhibit a mechanism of transition from $-\underline{1}$ to $+\underline{1}$ involving an energy barrier smaller than $H(\underline{0})-H(-\underline{1})$.

This is very easy to achieve if the volume Λ is sufficiently large.

In this paper we analyzed in detail the regions II and III, which happen to be, in a sense, the most interesting ones. In region IV one has the same local minima for the energy as in regions II and III; they are sets of noninteracting plurirectangles; but now the comparison between the times t_1, t_2, t_3, t_4 introduced in (3.13) changes totally. The main difference w.r.t. the regions II, III is that now, in IV, we have

$$M^*<L^*$$

and so we cannot even consider a possible mechanism of nucleation along a sequence of frames. Indeed one has that a birectangle is supercritical if and only if the minimal external side is not smaller than M^*. Then, as in region III but in a much easier way, we can prove that the escape from $-\underline{1}$ starts with an Ising-like nucleation of a protocritical droplet $\mathscr{P}_2$ leading to $\underline{0}$. But now, contrary to region III, the typical time $T^{-\underline{1} \to \underline{0}}$ for going from $-\underline{1}$ to $\underline{0}$ is much shorter than the typical time $T^{\underline{0} \to +\underline{1}}$ for going from $\underline{0}$ to $+\underline{1}$, so that the asymptotics of the time $T^{-\underline{1} \to +\underline{1}}$ of the transition from $-\underline{1}$ to $+\underline{1}$ is dominated by $T^{\underline{0} \to +\underline{1}}$.

The situation in which *a priori* one could expect a competition between the two metastable phases would be at first glance the union of the regions II, III, and IV. By arguing more carefully with a heuristic analysis of the heights of the possible barriers between $-\underline{1}$ and $\underline{0}$ and between $-\underline{1}$ and $+\underline{1}$ (given by the energy of formation of suitable critical droplets) one is led to expect that the two metastable phases corresponding to $-\underline{1}$ and $\underline{0}$ are in a sense really competing only around the half-line $0 < h = 2\lambda$ separating the regions II and III. This value $h = 2\lambda$ depends on the particular form of the Blume–Capel Hamiltonian.

The main result of the present paper is a rigorous proof of the above heuristics.

From the mathematical point of view we had to solve some large-deviation problems. This kind of problem would be extremely hard for a general nonreversible dynamics, but the treatment is very much simplified by the reversibility property of the dynamics.

In particular, to get the result we had to solve the minimax problem of the determination of the global saddle between $-\underline{1}$ and $+\underline{1}$. This is the really hard point of the work. We could handle the large-deviation problems à la Freidlin–Wentzell arising in the study of some rare events in the framework of our low-temperature Metropolis dynamics by taking advantage of a general approach to the study of typical trajectories, during the first exit from a non-completely attracted domain, recently developed in ref. 15. Nevertheless we still had to face the crucial model-dependent part consisting in solving some geometrically quite involved variational problems.

In particular we had to exclude, as highly reduced in probability, any mechanism of transition based on coalescence and we had to single out, among many others, only very few possible mechanisms of nucleation.

We were able to compute rigorously the lifetime of the metastable state, namely the typical transition times $T_{\lambda, h}$, for different values of the parameters λ, h. It turns out that these transition times are given by

$$T_{\lambda, h} \sim \exp(\beta \Gamma_{\lambda, h})$$

where the "activation energy" for very small values of λ, h has the expression

$$\Gamma_{\lambda, h} \sim \frac{8J^2}{h} \qquad \text{for} \quad 0 < \lambda < h < 2\lambda \tag{8.1}$$

and

$$\Gamma_{\lambda,h} \sim \frac{4J^2}{h-\lambda} \qquad \text{for} \quad 0 < 2\lambda < h \tag{8.2}$$

The value $4J^2/(h-\lambda)$ is just the activation energy $\Gamma_{\lambda,h}^{-\underline{1} \to \underline{0}}$ for the transition between $-\underline{1}$ and $\underline{0}$. The activation energy for the transition between $\underline{0}$ and $+\underline{1}$ is always (approximately) given by

$$\Gamma^{\underline{0} \to +\underline{1}} \sim \frac{4J^2}{h+\lambda} \tag{8.3}$$

In region III we have $\Gamma_{\lambda,h}^{-\underline{1} \to \underline{0}} > \Gamma_{\lambda,h}^{\underline{0} \to +\underline{1}}$ and this is the reason for (8.2); but in region IV we have the opposite, $\Gamma_{\lambda,h}^{-\underline{1} \to \underline{0}} < \Gamma_{\lambda,h}^{\underline{0} \to +\underline{1}}$, so that we get

$$\Gamma_{\lambda,h} \sim \frac{4J^2}{h+\lambda} \qquad \text{for} \quad 0 < -\lambda < h \tag{8.4}$$

This answers a question raised in ref. 6 about the "validity of Van't Hoff–Arrhenius law," which would predict in our case a decay $-\underline{1} \to \underline{0} \to +\underline{1}$ with an asymptotics of the transition time determined by $T^{-\underline{1} \to \underline{0}}$.

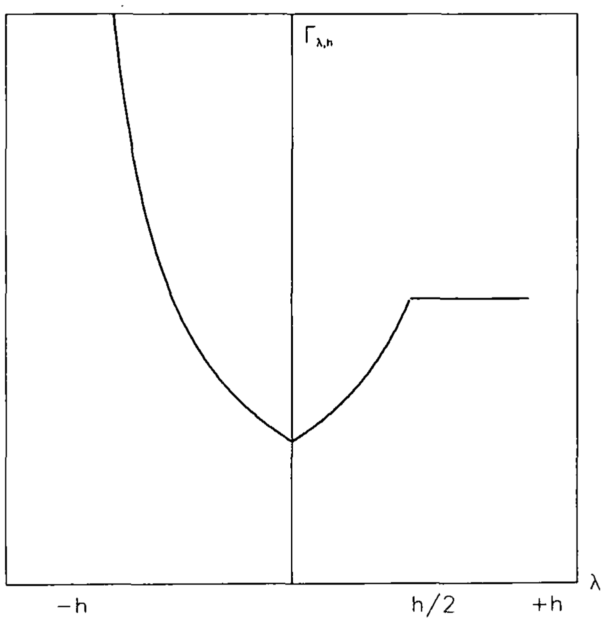


Fig. 46.

Our results can be interpreted by saying that this law is valid in the region III, whereas it is violated in regions II and IV for different reasons.

The new phenomenon about which apparently there is no reference even in the physics literature is the possibility of a "direct" transition between $-\underline{1}$ and $+\underline{1}$ and also a possible change in the mechanism of transition for different values of the parameters.

Notice that if we take a fixed small value of h and we vary λ, the analytic expression of $\Gamma_{\lambda,h}$ changes when we cross the lines $h = 2\lambda$, $\lambda = 0$.

We draw in Fig. 46 the graph of $\Gamma_{\lambda,h}$ as a function of λ for a fixed small value of h.

APPENDIX

In this appendix we want to state and prove Proposition A1 below. It refers to the first escape from a transient cycle A (see below) and, roughly speaking, it says that, under general hypotheses, with high probability, after many attempts, soon or later our process will *really* escape from A entering into different cycle by passing through one of the minimal saddles of the boundary of A.

The time for this transition has about the same asymptotics as the first hitting time to the boundary of A.

The result of Proposition A1 was used before without an explicit proof (see, e.g., refs. 9 and 10; it is, in fact, a simple consequence of the strong Markov property, but we think it useful, in order to better explain its statement, to eventually provide an explicit proof.

We will state our results in a slightly more general setup than the one considered in the present work (we also use a different notation). we will consider general Metropolis reversible Markov chains.

We suppose we are given an ergodic, aperiodic Markov chain $(X_t)_{t=0,1,2,\ldots}$ with finite state space Ω and with transition probabilities $P(x, y)$ satisfying the following hypothesis.

Hypothesis M. There exists a function $H: \Omega \to \mathbf{R}^+$ such that

$$P(x, y) = q(x, y) \exp(-\beta[H(y) - H(x)]_+) \tag{A.1}$$

where $q(x, y) = q(y, x)$ and $(a)_+$ is the positive part $(:= \max\{a, 0\})$ of the real number a.

The above choice corresponds to a *Metropolis Markov chain* which is *reversible* in the sense that

$$\forall x, x' \in \Omega: \mu(x)\, P(x, x') = \mu(x')\, P(x', x) \tag{A.2}$$

with

$$\mu(x) \propto \exp(-\beta H(x)) \tag{A.3}$$

One can introduce the notions of pair of communicating states, path, connected subset of Ω, boundary ∂Q of set $Q \subset \Omega$ cycles,... as the obvious generalizations of the corresponding ones given in Section 2.

For any set $Q \subset \Omega$ we introduce the set of all the minima of the energy in the boundary ∂Q of Q:

$$U(Q) := \{z \in \partial Q \colon \min_{x \in \partial Q} H(x) = H(z)\} \tag{A.4}$$

By $F(Q)$ we denote the set of the absolute minima of the energy in the set $Q \subset \Omega$:

$$F(Q) := \{y \in Q \colon \min_{x \in Q} H(x) = H(y)\} \tag{A.5}$$

A cycle A for which there exists $y^* \in U(A)$ *downhill connected* to some point x in A^c [namely $\exists x \notin A$, communicating with y^*, with $H(x) < H(y^*) =: H(U(A))$], is called *transient*; points like y^* are called (*minimal*) *saddles*. $\mathcal{S}(A)$ will denote the set of all minimal saddles of A.

Let $R = R(A)$ be the subset of A to which some point in $\mathcal{S}(A)$ is downhill connected:

$$R(A) := \{y \in A \text{ such that } \exists z \in \mathcal{S}(A) \text{ with } P(z, y) > 0\} \tag{A.6}$$

Let $V = V(A)$ be the analog of R outside A:

$$V(A) := \{y \notin A \text{ such that } \exists z \in \mathcal{S}(A) \text{ with } P(z, y) > 0\} \tag{A.7}$$

We set

$$\mathcal{H} := R(A) \cup V(A) \tag{A.8}$$

Proposition A.1. Consider a transient cycle A. Given $\varepsilon > 0$, let

$$T(\varepsilon) := \exp \beta[H(\mathcal{S}(A)) - H(F(A)) + \varepsilon] \tag{A.9}$$

Then, for every $\varepsilon > 0$, $x \in A$,

$$\lim_{\beta \to \infty} P_x(\tau_{(A \cup \partial A)^c} > T(\varepsilon)) = 0 \tag{A.10}$$

and

$$\lim_{\beta \to \infty} P_x(X_{\tau_{(A \cup \partial A)^c}} \in V(A)) = 1 \tag{A.11}$$

Proof. From Hypothesis M and the definition of $\mathscr{S}(A)$ we know that there exists a positive constant $c>0$, independent of β, such that

$$\inf_{x\in\mathscr{S}(A),\, y\in\mathscr{H}} P(x,y)>c, \qquad \lim_{\beta\to\infty}\ \sup_{x\in\mathscr{S}(A),\, y\notin\mathscr{H}} P(x,y)=0 \tag{A.12}$$

We define the sequence τ_i of stopping times corresponding to subsequent passage of our chain X_t in ∂A:

$$\begin{aligned}\tau_o &:= \inf\{t\geqslant 0: X_t\in\partial A\}\\ \sigma_1 &:= \inf\{t>\tau_o: X_t\notin\partial A\}\end{aligned} \tag{A.13}$$

and for $j=1,2,...$

$$\begin{aligned}\tau_j &:= \inf\{t>\sigma_j: X_t\in\partial A\}\\ \sigma_j &:= \inf\{t>\tau_{j-1}: X_t\notin\partial A\}\end{aligned} \tag{A.14}$$

We set, for $j=1,2,...,$

$$\mathscr{I}_j=[\tau_{j-1}+1,\tau_j], \qquad T_j:=|\mathscr{I}_j|=\tau_j-\tau_{j-1}-1 \tag{A.15}$$

Suppose that $X_{\tau_{j-1}+1}\in A$; let

$$\sigma_j^* := \min\{t>\tau_j: X_t\neq X_{\tau_j}\} \tag{A.16}$$

We say that the interval $\mathscr{I}_j$ is *good* if the following conditions are satisfied:

$$T_j<T(\varepsilon)$$

$$X_{\tau_j}\in\mathscr{S}(A)$$

$$X_{\sigma_j^*}\in\mathscr{H}$$

Let

$$j^* := \min\{j: T_j \text{ is not good}\}$$

Given the integer N, we want to estimate, for every $x\in A$, the probability $P_x(\tau_{V(A)}>NT(\varepsilon))$.

We write

$$\begin{aligned}P_x(\tau_{V(A)}>NT(\varepsilon)) &= P_x(\tau_{V(A)}>NT(\varepsilon); j^*>N)\\ &\quad + P_x(\tau_{V(A)}>NT(\varepsilon); j^*\leqslant N)\end{aligned} \tag{A.17}$$

Let us consider the first event in the decomposition given in (A.17): $\{\tau_{V(A)}>NT(\varepsilon); j^*>N\}$.

We have

$$\begin{aligned} &P_x(\tau_{V(A)} > NT(\varepsilon);\ j^* > N) \\ &\quad \leqslant P_x(X_{\tau_1} \in \mathscr{S}(A);\ X_{\tau_1+1} \notin V(A),\dots,\ X_{\tau_N} \in \mathscr{S}(A);\ X_{\tau_N+1} \notin V(A)) \\ &\quad \leqslant (1-c)^N \end{aligned} \tag{A.18}$$

Now, from Proposition 3.7 of ref. 15, (A.12), and the stationarity of our Markov chain we have

$$P_x(\mathscr{I}_j \text{ is not good}) \leqslant \delta(\beta) \tag{A.19}$$

with

$$\lim_{\beta \to \infty} \delta(\beta) = 0 \tag{A.20}$$

From (A.20) we get

$$P_x(\tau_{V(A)} > NT(\varepsilon);\ j^* \leqslant N) \leqslant P_x(j^* \leqslant N) \leqslant \delta(\beta)\, N \tag{A.21}$$

To conclude the proof it suffices to choose

$$N = N(\beta) = 1/\delta(\beta)$$

ACKNOWLEDGMENTS

We acknowledge R. Schonmann and especially Ch. Gruber for interesting discussions. We want to express thanks to Istituto Nazionale di Fisica Nucleare-Sezione di Bari, whose financial support made possible this collaboration. This work has been partially supported by grants CHRX-CT93-0411 and CIPA-CT92-4016 of the Commission of European Communities.

REFERENCES

1. M. Blume, *Phys. Rev.* **141**:517 (1966).
2. J. Bricmont and J. Slawny, *J. Stat. Phys.* **54**:89 (1989).
3. H. W. Capel, *Physica* **32**:96 (1966); **33**:295 (1967); **37**:423 (1967).
4. M. Cassandro, A. Galves, E. Olivieri, and M. E. Vares, Metastable behavior of stochastic dynamics: A pathwise approach, *J. Stat. Phys.* **35**:603–634 (1984).
5. I. Dinanburg and A. E. Mazel, *Commun. Math. Phys.* **125**:27 (1989).
6. T. Fig, B. M. Gorman, P. A. Rikvold, and M. A. Novotny, Numerical transfer-matrix study of a model with competing metastable states, *Phys. Rev. E* **50**:1930 (1994).
7. M. I. Freidlin and A. D. Wentzell, *Random Perturbations of Dynamical Systems* (Springer-Verlag, Berlin, 1984).

8. S. N. Isakov, Nonanalytic feature of the first order phase transition in the Ising model, *Commun. Math. Phys.* **95**:427–443 (1984).
9. R. Kotecky and E. Olivieri, Droplet dynamics for asymmetric Ising model, *J. Stat. Phys.* **70**:1121–1148 (1993).
10. R. Kotecky and E. Olivieri, Shapes of growing droplets—A model of escape from a metastable phase, *J. Stat. Phys.* **75**:409–507 (1994).
11. O. Lanford and D. Ruelle, Observable at infinity and states with short range correlations in statistical mechanics, *Commun. Math. Phys.* **13**:194–215 (1969).
12. F. Martinelli, E. Olivieri, and E. Scoppola, Metastability and exponential approach to equilibrium for low temperature stochastic Ising models, *J. Stat. Phys.* **61**(5/6):1105 (1990).
13. E. J. Neves and R. H. Schonmann, Critical droplets and metastability for a Glauber dynamics at very low temperatures, *Commun. Math. Phys.* **137**:209 (1991).
14. E. J. Neves and R. H. Schonmann, Behaviour of droplets for a class of Glauber dynamics at very low temperatures, *Prob. Theory Related Fields* **91**:331 (1992).
15. E. Olivieri and E. Scoppola, Markov chains with exponentially small transition probabilities: First exit problem from a general domain—I. The reversible case, *J. Stat. Phys.*, in press.
16. E. Olivieri and E. Scoppola, Markov chains with exponentially small transition probabilities: First exit problem from a general domain—II. The general case, In preparation.
17. O. Penrose and J. L. Lebowitz, Towards a rigorous molecular theory of metastability, in *Fluctuation Phenomena*, 2nd ed., E. W. Montroll and J. L. Lebowitz, eds. (North-Holland, Amsterdam, 1987).
18. O. Penrose and J. L. Lebowitz, Molecular theory of metastability: An update, Appendix to Towards a rigorous molecular theory of metastability, in *Fluctuation Phenomena*, 2nd ed., E. W. Montroll and J. L. Lebowitz, eds. (North-Holland, Amsterdam, 1987).
19. R. H. Schonmann, The pattern of escape from metastability of a stochastic Ising model, *Commun. Math. Phys.* **147**:231–240 (1992).
20. R. H. Schonmann, Slow droplet-driven relaxation of stochastic Ising models in the vicinity of the phase coexistence region, *Commun. Math. Phys.* **161**:1–49 (1994).
21. S. Shlosman and R. H. Schonmann, Preprint UCLA (1994).